Analyzing Power in Elastic Scattering of Nucleons off Nuclei

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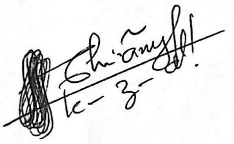
Abstract

Elastic scattering of nucleons off nuclei is an important tool in learning about distributions of nucleons, protons and neutrons, within nuclei. Experimentally, differential cross sections are measured as a function of scattering angle and, for polarized nucleons, analyzing power. Theoretically, scattering phase shifts are computed by solving a set of radial Schrodinger equations, assuming an optical potential for the nucleons. The goal of this project is to arrive at simple formulas allowing to compute analyzing power from the phase shifts, in preparation for the global fit of optical potential parameters to elastic and quasielastic data on nucleonic scattering.

Keywords: Elastic scattering; Coulomb potential; Polarized particle; differential cross section; analyzing power.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



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1. Introduction

1.1 General Background

Deviations of a particle's trajectory from its original direction are observed when particle toward another no matter whether the two are in a direct contact or not. The same observation has been seen with radiation, light or sound. This implies that there should be a certain potential on the particle's trajectory, whether certain interaction forces act which can be electromagnetic, electrostatic and Coulomb. From a macroscopic point of view, two bodies in motion can collide with the collision bringing each body to a certain new direction. From a microscopic point of view, particles such as molecules, atoms, electrons and photons, having some properties, move with certain energy in different directions and collide while influencing each other through interaction forces. During the interaction, some of the particles are absorbed and others are redirected with some probability. This introduces the concept of scattering particles, through which particle's properties are changed or remain conserved.

In laboratories, physicists started doing some experiments on collision and scattering of particles. The goal of these experiments was to probe the properties of particles and the kind of interactions occurring between them during the experiment. Some of the results from these experiments are the discovery of the atomic model by Ernest Rutherford in 1909, who used charged particles in an elastic scattering from a gold nuclei. Another result is the evaluation of radii of nuclear distribution using protons (De Vries et al., 1987), alphas (Gils et al., 1984) and pions in elastic scattering off nuclei (Friedman, 2012). Today, the scattering process plays a great role in microscopic physics. In nuclear reaction where two nuclei collide, incident and target, various processes such as nuclear excitations are induced.

The main measurable parameters from a scattering experiment are the scattering angle, the differential cross section, the analyzing power and the spin rotation among the characteristics of particles involved in the process. The scattering angle gives the direction of a scattered particle and the differential cross section represents the ratio of flux of scattered particles collected into a certain solid angle over flux of incident particles. The analyzing power gives a fractional difference in cross section for scattering to the left versus right for particles that are initially polarized. The differential cross section are measured as a function of scattering angle. For polarized particles the analyzing power is also measured as function of scattering angle. The collection of the experiment values of these parameters are called scattering data. In many laboratories through the world, experiments are done, and scattering data are growing and this becomes the work of most of the nuclear physicists.

1.2 Problem statement

An important work which interests nuclear physicists is to derive a mathematical model able to compute the same scattering parameters for a given process and argue the ability of the theoretical expressions to describe the experimental scattering results. An example is the work of Uesaka et al. (2010) who investigated the sensitivity, the reaction and interaction of nuclear structure by constructing a T-matrix model which could predict the analyzing power and fits the experimental elastic scattering of protons from neutron-rich ^6He nucleus. Cooper and Horowitz (2005) have calculated in 2005 the vector analyzing power for elastic electron scattering from a variety of spin zero nuclei at energies from 14 MeV to 3 GeV.

The choice of range of incident particles energy and the kind of potential present during the scattering process has pushed the research into advanced areas. In the work of Koning and Delaroche (2003) an optical model potential for neutrons and protons with incident energies from 1 keV up to 200 MeV has been presented for spherical nuclides in the mass range A [24;209]. The model requires a numerical solution of the Schrödinger equation in order to predict of the basic scattering observable parameters. By modifying a little the Koning optical model potential, Danielewicz et al. (2017) analysed only differential cross section for elastic and quasi-elastic reactions, following the concepts of isoscalar and isovector potentials combined into a Lane potential in order to detect and quantify probable displacement of the isovector and isoscalar surfaces predicted by macroscopic considerations and by computations.

This present study aims at a simple mathematical formula for calculating both differential cross section and analyzing power for an elastic scattering of nucleons off nuclei under a combined Coulomb and nuclear potential. The derivation is simplified by considering the asymptotic form of the expansion of solution to a set of radial Schroödinger equations under assumption of spherical symmetry and simply extract the expression of scattering amplitude needed to express the differential cross section and analyzing power. In this way, there is not need to solve any Schrödinger equation numerically in looking for the expression.

1.3 Objectives

The main objective of this work is to derive an expression for the analyzing power in the simple form for scattering of polarized nucleons off a nuclei. To reach this main objective, we first describe the scattering theory, we establish the time-independent Schrödinger differential equation which describes the motion of particles during the scattering process under some assumptions: Incident particles are initially considered without charge and spin and the same for the target. For these we derive the differential cross section only. Next we consider incident particles with spin $+\frac{1}{2}$ and target without spin and end up with expressions for the differential cross section and analyzing power. As the last case we consider incident particle with spin and charge so that the reaction occurs under combined effect of Coulomb and nuclear potential. The phase shifts for nuclear reactions are theoretically computed by solving a set of radial Schrödinger differential equation, but this will not be the case in this work. The phases will appear within the expressions for differential cross section and analyzing power and a later work will be concerned specially one how to compute the phase shifts.

1.4 Significance

The significance of this work is related to the later work concerning in how to use the simple mathematical formulas to which scattering data are applied in order to compute the theoretical analyzing power and differential cross section. In fact, a typical scattering process characterized by incident particle energy is considered. Experimental differential cross section and the analyzing power function of the scattering angle can be measured and used to be compared to the theoretical values in order to analyse existing data and to predict the outcomes of a future measurements.

1.5 Layout of this work

The layout of this work is as follows: Chapter 2 presents the general scattering theory from classical mechanics to quantum mechanics perspective, and describes essence of scattering experiment and its measurements. Chapter 3 describes the derivation of the expressions to differential cross section for spinless and uncharged and charged particles. This is followed by Chapter 4 describing how analyzing power and differential cross section are derived from an elastic scattering of spin charged incident particle from spinless target under Coulomb and nuclear short-ranged potentials. Finally, Chapter 5 summarizes the conclusions of this study.

2. Generality on the scattering theory

2.1 Scattering process

Scattering process is a real physical process which occurs when some forms of moving particles are forced to deviate and/or to reflect from a straight line by one or more paths due to the localized non-uniformities in the medium through which they pass. It is a type of collision of particles such as between molecules, atoms, electrons and nucleons. Examples of this phenomena include the *cosmic ray* scattering in the Earth's upper atmosphere, the *particle collisions* inside particle accelerators, the *electron scattering* by gas atoms in fluorescent lamps and the *neutron scattering* inside nuclear reactors. Scattering can be caused by non-uniformities, known as scatterers or scattering centers, or another particle. The reflection, the refraction, and the diffraction etc. are all forms of scattering.

There are different types of scattering process: the *elastic* scattering which occurs while the wavelength (frequency) of the scattered radiation is the same as the incident radiation, the *inelastic* scattering which occurs while the emitted radiation has a wavelength different from that of the incident radiation, the *quasi-elastic* scattering occurs while the wavelength (frequency) of the scattered radiation shifts, the *single* scattering which occurs while the photons occurs only once during the process and the *multiple* scattering which occurs when the photons can be scattered many times before emerging.

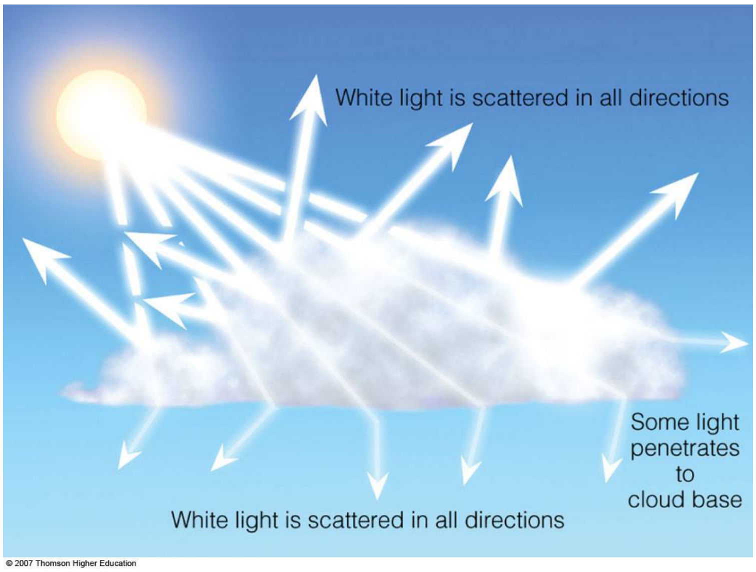

Figure 2.1: Mie Scattering that occurs in the Atmosphere.
Source : apollo.lsc.vsc.edu

The common natural scattering processes seen in the atmosphere are the *Rayleigh Scattering*[1] and the *Mie scattering*[2] shown on the Fig. 2.1, which occurs when sunlight reaches the Earth, and is filtered through the atmosphere before hitting the surface.

In physics, for the particles involved in the phenomena, the scattering process can be explained in

[1] Scattering of light off air molecules which involves particles much smaller than the wavelength of incident light. It is responsible for the blue color of clear sky

[2] An elastic scattering off molecules that have a diameter similar to or larger than the wavelength of the incident light

classical point of view and also quantum point of view.

2.2 Classical scattering process

Consider an electron which scatters off a fixed atom, and moving with in some approximately straight orbit and a complicated orbit during the interaction time. Typically, we observe the free motion of the particle before and after the collision and ignore the event in the neighbourhood of the target atom.

From Newton's second law, the free motion of the particle is described by :

$$M\ddot{x}(t) = -\nabla V(x) \tag{2.2.1}$$

where $\ddot{x}(t)$ is the acceleration of particle of mass M while scattering from the potential $V(x)$. Since we are regarding to the asymptotical behavior of the orbit, so we look at the motion at t goes to $-\infty$ for the incident particle and t goes to $+\infty$ fot the scattered particle. The key aspects of the actual orbit $x(t)$, solution to the equation (2.2.1), are the asymptotic characteristics

before

$$x_{in}(t) = x_{in} + v_{in} \cdot t \qquad\qquad t \longrightarrow -\infty \tag{2.2.2}$$

after interaction

$$x_{out}(t) = x_{out} + v_{out} \cdot t \qquad\qquad t \longrightarrow -\infty \tag{2.2.3}$$

where $x_{in}(t)$ and $x_{out}(t)$ are called the incoming and outgoing asymptotes of the actual scattering orbit $x(t)$. The complete solution to this scattering problem can be solved by knowing first all the asymptote orbits.

2.3 Quantum scattering process

The quantum scattering process is similar to the classical process, but in this case the orbit $x(t)$ is replaced by a state vector $|\psi(t)\rangle$ which satisfies the time dependent Schrödinger equation (2.3.1) as the orbit $x(t)$ satisfies Newton's second law in the classical process.

$$i\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \tag{2.3.1}$$

The solution to this equation (2.3.1), classically considered as an orbit, is of the form

$$|\psi(t)\rangle = U(t)|\psi\rangle = e^{-iHt}|\psi\rangle \tag{2.3.2}$$

where $U(t)$ is an evolution operator and $|\psi\rangle$ any vector in the corresponding Hilbert space $\mathcal{H}$, and H is the Hamiltonian.

For an illustration, we assume that the particle is spinless, as it is explained in the next section. Similar to the classical process, the particle is within a local potential $V(x)$ function of the particle's position. The solution (2.3.2) describes the evolution of the scattering process. It represents a free wave packet localized far away from the target particle. Its motion is given by $U_0(t)|\psi_{in}\rangle$ as t goes to $-\infty$ and $U_0(t)|\psi_{out}\rangle$ as t goes to $+\infty$ after collision, i.e.

$$U(t)|\psi\rangle = U_0(t)|\psi_{in}\rangle \qquad \texttt{for} \qquad t \longrightarrow -\infty \tag{2.3.3}$$

$$U(t)|\psi\rangle = U_0(t)|\psi_{out}\rangle \qquad \texttt{for} \qquad t \longrightarrow +\infty \tag{2.3.4}$$

where $|\psi_{in}\rangle$ and $|\psi_{out}\rangle$ are the asymptotic free orbit.

There are some conditions satisfied by the potential $V(x)$ for the asymptotic behaviour of an orbit to occur in the scattering process. For a spherical potential $V(r)$, the conditions are such that $V(r)$ falls off quicker than r^{-3} at ∞, $V(r)$ is less singular than $r^{-3/2}$ at the origin and $V(r)$ is reasonably smooth in between.

2.4 Particles in scattering experiment

The particles involved in the scattering process are commonly elementary. Whether it is the incident or the target, particles have some measurable properties. A particle can be described by its *mass* (m), its the *charge* (q), *lifetime* and some *quantum numbers*. Quantum numbers can be set of integers or half-integers which describe specially the energy levels $(n = 1, 2, 3 \cdots)$, the azimuthal quantum number or orbital angular momentum (l) which defines the magnitude of the angular momentum trough the relation $L^2 = \hbar^2 l(l+1)$, the magnetic quantum number (m_l) or the projection of the orbital angular momentum along a specified axis such that $l_z = m_l\hbar$ and the spin angular momentum which defines the intrinsic angular momentum within the orbital. Fig. (2.2) below illustrates each quantum number of a particle.

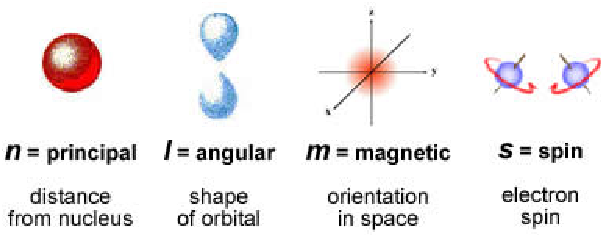

Figure 2.2: Four quantum numbers of particles: n, l, m, and s. .
Source : www.micalex.com

Usually, in a scattering experiment, the incident particles are coming in with the above properties and affect the scattering observables during the experiment explained in the next section. In addition, the spin angular momentum have two states: spin $s = +1/2$ (or spin-up) and spin $s = -1/2$ (or spin-down). It is not easy to determine precisely the orientation of the particles when there are incident on a target. This leads to the concept of *unpolarized* and *polarized* particles.

In the scattering experiment, one often measures the cross sections with no regard for spin of the particles. However, particles can be characterized by the spin state. Incident beam can contain unpolarized particles with spin-up and down, also can contain one type of spin through the use of a polarizer. When particle has spin, we can determine the analyzing power, a physical quantity that quantifies the polarization of a particle during a scattering process. The polarizer collects unpolarized particles from one side and lets pass the particles with spin-up or spin-down only, as shown in the below Fig. (2.3)

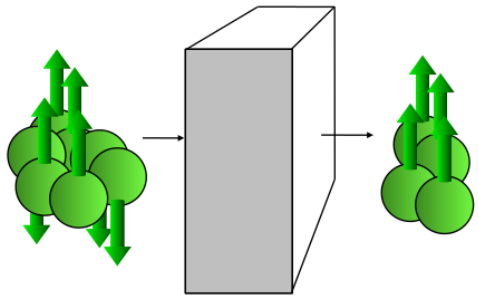

Figure 2.3: The polarizer receives particles on the left side with upward and downward spin and transmits only spin up particles to the right. Source : Stewart (2011)

In this work, the incident particles considered are the nucleons, with spin angular momentum 1/2 elastically scattered from a spinless nuclei.

In quantum mechanics, generally, the angular momentum is an operator and is defined by:

$$\vec{J} \equiv \vec{r} \times \vec{p} \tag{2.4.1}$$

where r and p are respectively position and momentum vector operators of a given particle. The angular momentum satisfies commutation rules below:

$$\begin{cases} [J_x, J_y] = i\ J_z \\ [J_y, J_z] = i\ J_x \\ [J_z, J_x] = i\ J_y \end{cases} \quad \text{and} \begin{cases} J^2 = J_x^2 + J_y^2 + J_z^2 \\ [J, J^2] = 0 \end{cases} \tag{2.4.2}$$

2.4.1 Definition (Eigenvector and eigenvalues of the angular momentum). (Messiah, 1962): The eigenvector of the square angular momentum J^2 is a state of the angular momentum (jm), denoted by $|jm\rangle$. The conditions imposed on $|jm\rangle$ are such that:

$$J^2|jm\rangle = j(j+1)|jm\rangle \qquad \text{and} \qquad J_z|jm\rangle = m|jm\rangle \tag{2.4.3}$$

where $j(j+1)$ and m are respectively eigenvalue of J^2 and J_z, with $-j \leq m \leq +j$ as described in the Appendix (A.2.5) and $j = 0, 1/2, 1, 3/2. \cdots, \infty$. From there, we can deduce that if $J^2 = j(j+1)$ and m correspond to the same eigenstate $|jm\rangle$ then the above are only possible values of m .

2.4.2 Definition (Eigenvector and eigenvalues of the orbital angular momentum). (Messiah, 1962): Similarly for the orbital angular momentum l, the eigenvalues of the square angular momentum l^2 and l are respectively $l(l+1)$ and m_l. The eigenfunction of l is a function of the angle (θ, ϕ) denoted by $F_l^m(\theta, \phi)$ such that

$$l^2\ F_l^m(\theta,\phi) = l(l+1)\ F_l^m(\theta,\phi) \qquad \text{and} \qquad l_z\ F_l^m(\theta,\phi) = m_l\ F_l^m(\theta,\phi) \tag{2.4.4}$$

As shown in the appendix (A.4.13), the common eigenfunctions of l^2 and l_z, corresponding to the eigenvalues (lm) represent an orthonormal complete set of functions θ and ϕ denoted by $Y_l^m(\theta, \phi)$. They are given by

$$Y_{lm}(\theta, \phi) = \begin{cases} (-1)^m \left[\dfrac{2l+1}{4\pi} \dfrac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(\cos\theta) e^{im\phi} \text{ for } m \geq 0 \\ \\ (-1)^m Y_{lm}^*(\theta, \phi) \text{ for } m < 0 \end{cases} \tag{2.4.5}$$

For $m = 0$, the eigenfunction is the normalized Legendre polynomial

$$Y_l^0(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(cos\ \theta) \tag{2.4.6}$$

Usually, the particles in the scattering process, are moving with more than one angular momentum, In quantum mechanics, the Hamiltonian operator remains invariant under rotation and commute with all the component of the total angular momentum. Therefore, for given particles, with each of them angular momenta J_1 and J_2, we need to find the total angular momentum J given by

$$J = J_1 + J_2 \tag{2.4.7}$$

Each angular momentum operates in its own space or eigenstate $|j_i m_i\rangle$, $i = 1, 2$. Generally, the eigenstates $|j_1 m_1\rangle$ and $|j_2 m_2\rangle$ are not the same, therefore we need first to define a common eigenstate $|j_1 m_1 j_2 m_2\rangle$ where both angular momenta are operating, in order to obtain a new operator J which is the total angular momentum of the particle. Once we know the total angular momentum and its eigenstate, we now determine its eigenfunction which describes the motion of the particle in the scattering process.

As defined in the Eqn.(2.4.2), the eigenstate $|j_1 m_1 j_2 m_2\rangle$ is the same for J_1^2, J_2^2, $J_{1,z}$ and $J_{2,z}$.

Let defined the common eigenstate by $|m_1 m_2\rangle$ and the eigenstate of the total angular momentum by $|J\ M\rangle$.

2.4.3 Theorem (Fundamental addition theorem). *In the $(2j_1+1)(2j_2+1)$-dimensional space spanned by the eigenstate $|j_1 m_1 j_2 m_2\rangle$ with j_1, j_2 fixed and $m_1 m_2$ variables,*

(i) The possible values of J are

$$j_1 + j_2, j_1 + j_2 - 1, \cdots, |j_1 - j_2| \tag{2.4.8}$$

(ii) To each of these values there corresponds one and only one series of $(2J+1)$ eigenstate $|J\ M\rangle$. of the total angular momentum.

From there theorem 2.4.3, we can deduce that to each pair (JM) there corresponds an eigenstate $|j_1 j_2 J\ M\rangle$ of the total angular momentum J. This eigenstate is defined by a unitary transformation from one basis to other basis: (Messiah, 1962)

$$|j_1 j_2 J\ M\rangle = \sum_{m_1\ m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2 | j_1 j_2 J\ M\rangle \tag{2.4.9}$$

where $\langle j_1 j_2 m_1 m_2 | j_1 j_2 J\, M\rangle$ is called Clebsch-Gordan coefficients $C_{j_1 j_2}$ which can be written like

$$C_{j_1 j_2} = \langle j_1 j_2 m_1 m_2 | J\, M\rangle \tag{2.4.10}$$

The explicit form of the Clebsch-Gordan coefficients is given in the Appendix (C.1).

2.5 Scattering experiment

Scattering experiment utilizes a beam of particles as electrons, protons, neutrons, muons, atoms, or quanta of electromagnetic radiations, emerging from a source of particles or a particle accelerator. The projectile particles p_i, characterized by some quantum numbers k_i and energy E_i, are projected toward a target particles p_t with also some quantum numbers k_t and net energy E_t. Within a collision region, interaction forces net yielding scattered particles p_s, going away from the target area with a certain quantum numbers k_s and energy E_s, and the modified target particles $p_{t,s}$ with quantum numbers $k_{t,s}$ and energy $E_{t,s}$.

At a certain position away from the target, a detector collects the scattered particles and measures the probability of a given particle scattering in a particular direction (θ, ϕ) called *scattering angle* around the target, as shown below in Fig. (2.4).

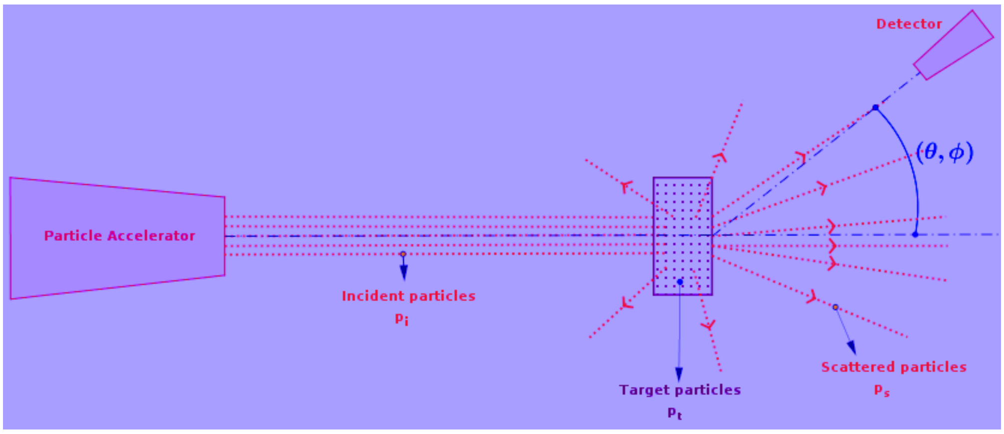

Figure 2.4: Schematic diagram of a scattering experiment. The particle accelerator is on the left, target in the middle, and detector on the right. Source: Design generated using Geogebra software

Schematically, the scattering process may be represented as

$$p_i + p_t \longrightarrow p_{s,t} + p_s \tag{2.5.1}$$

All the information about the scattering process is arrived at by examining the scattering particles collected by the detector. The information from the experiment form the scattering data.

For the elastic scattering as we explain later, the conserved additive quantum numbers follow the Equations (2.5.2) and (2.5.3) below:

$$k_i + k_t = k_{s,t} + k_s \tag{2.5.2}$$

and, in particular, for energy

$$E_i + E_t = E_{s,t} + E_s \tag{2.5.3}$$

For the experiments, there exist important classes of scattering configurations, discussed below.

- **Elastic scattering off immobile targets** occurs when the target is much heavier than the incident particle and its mass can be considered infinite practice. In this case, the scattering is completely elastic, i.e. the total energy is conserved ($\Delta E_i + \Delta E_t = 0$) but with a possibility for an arbitrary momentum exchange ($\Delta k \neq 0$). The target can be modelled by some potential fixed in real space.
- **Elastic scattering off general targets** is similar to that for immobile targets, but in this case, the target's mass is not assumed infinite.
- **Inelastic scattering** refers to the lost in energy of the incident particles in exciting the target particles. Exchange of energy occurs and the reaction can even produce a new element.
- **Quasi-elastic scattering** is defined by a low excitation energy of the target particles, and implies that the incident particle loses only a small fraction of its energy. (Goldberger and Watson, 1967)
- **Rearrangement scattering** involves a certain reaction of the target particles during the interaction which produces a new particle.
- **Resonance scattering** leads to the formation and subsequent decay of long lived particle. In nuclear physics, the importance of this process lies the fact that the scattering of nucleons off heavy nuclei is able to generate a long lived transient state. (Altland, 2005)

Scattering experiment is one of the important processes serving to extract information from a microscopic structure of quantum systems. It plays essential role in many fields such as atomic, nuclear, high energy, and condensed matter physics. For example, in atomic physics, the Rutherford's discovery of the nucleus came from his study of a Alpha particle scattering off a gold foil. The Franck-Hertz experiment established the existence of atomic energy levels by observation of electrons scattering off mercury vapour. In nuclear physics, the first clear evidence of nuclear structure came followings Rutherford's observations. (Taylor, 1972)

2.6 Observable parameters

In quantum scattering process, the observable parameters depend on the properties of the particles that scatter. Usually, for more information about the interaction between particles within the scattering region, the spin behaviour is taken in account. This implies two kind of observable the spin related and, spin independent observables. Spin independent work for with no spin and/or disregard spin and spin observables take in account the spin state of the incident and scattered particles and the spin sensitivity of the detector. (Landau, 2004)

The observable parameters are the differential cross section, the analyzing power and the spin rotation. Briefly we describe below each of them and in the next chapter we show how to derive their expressions.

2.6.1 Differential cross section (DCS).

Differential cross section is a quantity obtained from the scattering experiment, that provides most essential information about the properties of atomic and subatomic particles during their interaction in the scattering process.

To introduce the concept of this quantity we consider a beam of particles, moving towards a target along a z-axis. We define the number of the particles crossing an transverse area, in the (x, y)-plane, per unit of time as flux of incoming particles I. Let us now consider an elastic scattering process such that any incident particle becomes a scattered particle and no particles are absorbed or transformed by the target, and the energy is conserved. Practically, some particles will pass the target undeflected, but some will be scattered into other directions, characterized by the angles (θ, ϕ) relative to the z-axis and x-axis in the transverse plane, see Fig. (2.5). The particles are scattered in directions between $(\theta, \theta + d\theta)$ and $(\phi, \phi + d\phi)$.

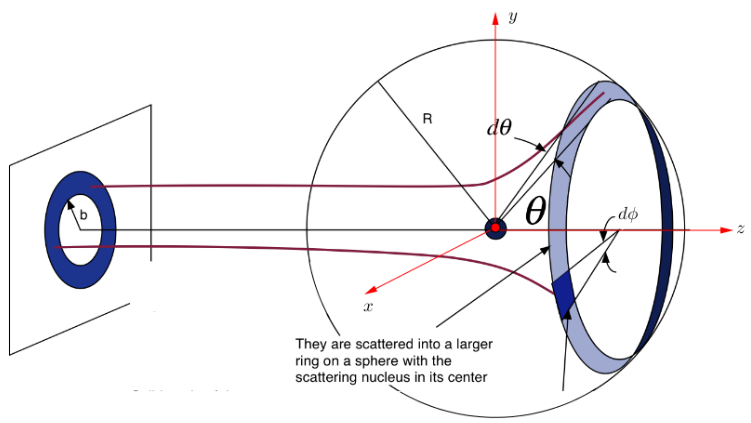

Figure 2.5: Pic of the scattering cross section in three dimensions. Source: Harr (2015)

The number of the scattered particles dN collected by the detector is proportional to the time of measurement t, the flux of incident particles I, the number of the number of the particles within the target N_t swept by the beam and the solid angle through which pass the scattered particles collected by the detector $d\Omega$. i.e:

$$dN(\theta, \phi) \propto t \cdot d\Omega \cdot I \cdot N_t \tag{2.6.1}$$

In the absence of external fields or other interactions, dN depends only on the scattering angle (θ, ϕ).

The Eqn. (2.6.1) can be written as

$$dN(\theta,\phi) = \frac{d\sigma}{d\Omega}(\theta,\phi)\ t\ d\Omega\ I\ n_a A \tag{2.6.2}$$

where n_a and A are respectively the density area and the area of the target through which pass the incident particles and $N_t = n_a A$, and $\dfrac{d\sigma}{d\Omega}$ is a coefficient of proportionality named differential cross section. We define $N_p = t\ I\ A$ as a number of incident particles coming to the target, therefore the number of scattered particles dN collected by the detector is given by

$$dN(\theta,\phi) = \frac{d\sigma}{d\Omega}(\theta,\phi)\ d\Omega N_p n_a \tag{2.6.3}$$

Therefore, the differential cross section can be expressed as

$$\frac{d\sigma}{d\Omega}(\theta,\phi) = \frac{dN(\theta,\phi)}{d\Omega N_p n_a} \tag{2.6.4}$$

where N_p, dN and $d\Omega$ are known numbers, n_a is in m^{-2}/Sr, and $\frac{d\sigma}{d\Omega}(\theta,\phi)$ in m^2. The order of magnitude of the differential cross section is $10^{-24}\ cm^2 = 1\ barn$. In the above, $\frac{dN}{d\Omega}$ is measured and, N_p and n_a are knowns.

The effective area of the target for scattering particles in any direction, known as the total cross section for elastic scatterings is obtained by integrating differential cross section over the full solid angle 4π:

$$\sigma(\theta,\phi) = \int\limits_{\Omega} \frac{d\sigma}{d\Omega} d\Omega = \int\limits_0^{2\pi} d\phi \int\limits_0^{\pi} d\theta \sin\theta \frac{d\sigma}{d\Omega} \tag{2.6.5}$$

For classical mechanics point of view, we can express σ in term of the impact parameter b as : Danielewicz (2016)

$$\sigma(\theta,\phi) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{2.6.6}$$

For the large value of the impact parameter b, this implies the small value of the scattering angle θ, and vice versa. If the incident particles occur with higher velocity or higher kinetic energy, they will not spend much time during the interaction so that their trajectory is a straight line and the scattering angle will be small. The relationship between b and θ will depend on the kind of the scattering potential present during the experiment.

In quantum mechanics, the formalism for treating the properties and interactions of the particles is different than in classical mechanics. In quantum scattering process, by considering first the spinless case, we consider an incident particle represented by a plane wave, see Fig. (2.6a) travelling in z direction and encountering a scattering potential that produces an outgoing spherical wave, see Fig.(2.6b):

(a) (b)

Figure 2.6: View of plane wave (a) and spherical wave (b) during a scattering process

The motion of a particle, affected by a certain scattering potential denoted as $V(\vec{r})$, is described by a time-independent Schrödinger equation given by

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right)\psi(\vec{r}) = E\psi(\vec{r}) \tag{2.6.7}$$

where E is the total energy of the particle and $\psi(\vec{r})$ is the wave function describing the motion of the particle incident on the scattering centre in the z-direction and scattered in the different directions (θ, ϕ). By considering the motion far away from the scattering potential, as described above, the asymptotic solution of Eqn. (2.6.7) is given by

$$\psi(r, \theta, \phi) = A_k\left(e^{ikz} + f_k(\theta, \phi)\frac{e^{ikr}}{r}\right) \tag{2.6.8}$$

where $f(\theta, \phi)$ is the scattering amplitude. The differential cross section is now determined by the probability for a particle in the initial plane wave to end up as a spherical wave in a certain direction (θ, ϕ).

Consider an infinitesimal volume dV of cross section $d\sigma$, the probability of an incident wave-function $\psi_i(r)$ passing through with a velocity v during a infinitesimal time dt is given by

$$d\mathbb{P} = |\psi_i(r)|^2\, dV \tag{2.6.9}$$

where $\psi_i(r)$ is given by the first term of the Eqn. (2.6.8) and $dV = v\; dt\; d\sigma$. The probability becomes

$$d\mathbb{P} = |A_k|^2 v\; dt\; d\sigma \tag{2.6.10}$$

This is the probability for the same incident wave-function to be found in a corresponding solid angle $d\Omega$ at a distance r from the target:

$$d\mathbb{P} = |\psi_{sc}(r)|^2\, dV \tag{2.6.11}$$

where $\psi_{sc}(r)$ is the scattered wave-function given by the second term of the Eqn. (2.6.8) and $dV = v\; dt\; r^2 d\Omega$.

so that

$$d\mathbb{P} = \left| \frac{A_k}{r} f_k(\theta, \phi) \right|^2 v \; dt \; r^2 d\Omega \tag{2.6.12}$$

By equality of the probability, the Eqn. (2.6.10) and (2.6.12) give

$$\frac{d\sigma}{d\Omega}(\theta, \phi) = |f_k(\theta, \phi)|^2 \tag{2.6.13}$$

where k is the wave number related to the energy of the incident particles and given by

$$k = \frac{\sqrt{2mE}}{\hbar} \tag{2.6.14}$$

The Jacobi-Anger identity, which expands an exponential of a trigonometric function in the basis of the harmonic functions, is useful here:

$$e^{iz} = \sum_{n=-\infty}^{\infty} i^n \; j_n(z) \; e^{in\theta} \tag{2.6.15}$$

with i the imaginary unit and j_n is the spherical Bessel function. With this, we can expand the incoming wave function as

$$e^{ikz} = \sum_{l=0}^{\infty} (2l+1) \; i^l \; j_l(kr) \; P_l(\vec{k} \cdot \vec{r}) \tag{2.6.16}$$

where i is the imaginary unit, $\vec{k}$ is the wave vector unit, $\vec{r}$ is the position unit vector such that $\vec{k} \cdot \vec{r} = \cos(\theta)$, j_l are spherical Bessel functions and P_l are Legendre polynomials defined in the Appendix (B).

2.6.2 Analyzing power.

Analyzing power is the most important observable in scattering process tied to spin and expressible in terms of which is the effect on the differential cross section when the incident particle is polarized.

The complete information about the scattering amplitude is obtained by considering the incident particles with all possible angular momenta. In scattering process for particle with spin, we combine the spaces of internal and orbital spins. Consider a incident particle in intrinsic spin state α. After a collision the scattered particle can be remain in the same spin state or to shift in the spin state β, and similarly for incident particle in spin state β. In this case the asymptotic wave function is given by (Rodberg and Thaler, 1967)

$$\langle \sigma | \psi_\alpha(r) \rangle = \langle \sigma | \alpha \rangle e^{ikz} + [f_\alpha(\Omega) \langle \sigma | \alpha \rangle + g_\alpha(\Omega) \langle \sigma | \beta \rangle] \frac{e^{ikr}}{r} \tag{2.6.17}$$

and

$$\langle \sigma | \psi_\beta(r) \rangle = \langle \sigma | \beta \rangle e^{ikz} + [f_\beta(\Omega) \langle \sigma | \alpha \rangle + g_\beta(\Omega) \langle \sigma | \beta \rangle] \frac{e^{ikr}}{r} \tag{2.6.18}$$

where $\Omega \equiv (\theta, \phi)$.

For unpolarized incident particles, we have a linear combination of both spin states given by

$$|\chi_i\rangle = a_\alpha|\alpha\rangle + a_\beta|\beta\rangle \tag{2.6.19}$$

The actual wave function is then the sum of (2.6.17) and (2.6.18)

$$\langle\sigma|\psi(r)\rangle = a_\alpha\langle\sigma|\psi_\alpha(r)\rangle + a_\beta\langle\sigma|\psi_\beta(r)\rangle \tag{2.6.20}$$

This leads to

$$\langle\sigma|\psi(r)\rangle = \langle\sigma|\chi_i\rangle e^{ikz} + \langle\sigma|M(\Omega)|\chi_i(r)\rangle\frac{e^{ikr}}{r} \tag{2.6.21}$$

where $M(\Omega)$ is the 2 $\times$ 2 matrix scattering amplitude given

$$M(\Omega) = \begin{pmatrix} f_\alpha(k,\Omega) & g_\alpha(k,\Omega) \\ g_\beta(k,\Omega) & f_\beta(k,\Omega) \end{pmatrix} \tag{2.6.22}$$

Under the invariance of rotation and parity and for scattering process of spin-half particles off a spin-zero target, with weak interaction, only two of the four functions matrix elements are independent. This implies that following equalities

$$f_\alpha(k,\Omega) = f_\beta(k,\Omega) = f(k,\theta) \tag{2.6.23}$$

$$g_\alpha(k,\Omega) = g(k,\theta)e^{-i\phi} \text{ and } g_\beta(k,\Omega) = -g(k,\theta)e^{i\phi} \tag{2.6.24}$$

Here, the azimuthal angle ϕ is relative to the plane spanned by $\vec{k}$ and $\vec{k}'$ and the spin directions are along or opposite to $\vec{k}$.

Then matrix scattering amplitude becomes

$$M(k,\Omega) = \begin{pmatrix} f(k,\theta) & g(k,\theta)e^{-i\phi} \\ -g(k,\theta)e^{i\phi} & f(k,\theta) \end{pmatrix} \tag{2.6.25}$$

Every 2 $\times$2 matrix can be written in term of the linear combination of a 2$\times$ 2 identity matrix and a 2$\times$2 Pauli matrix $\vec{\sigma}$. The above scattering matrix can be, in particular, expressed as

$$M(k,\theta,\phi) = f(k,\theta)\mathbb{I}_{2\times 2} + ig(k,\theta)\ \sigma\cdot\vec{n} \tag{2.6.26}$$

and

$$\vec{n} = \frac{k \times k'}{\|k \times k'\|} \tag{2.6.27}$$

where k and k' are respectively incident and scattered electron vectors as shown in the below Fig. (2.7)

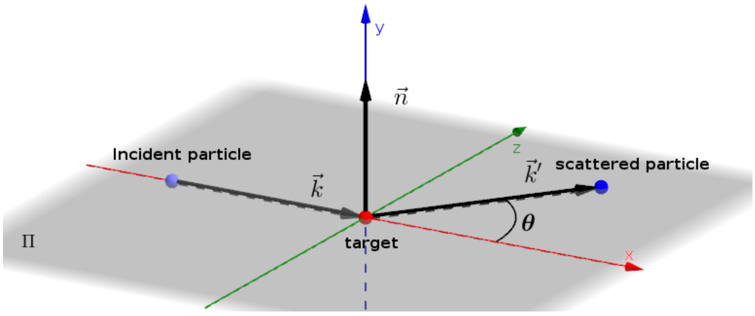

Figure 2.7: Scattering plane is spanned by the initial and final wave-vectors k and k'. Source:Drawn using Geogebra Software

Therefore, for incident particles polarized a long $\vec{k}$ the differential cross sections for retaining spin direction is

$$\left.\frac{d\sigma}{d\Omega}(\theta)\right|_{(\uparrow)} = |f(k,\theta)|^2 \tag{2.6.28}$$

and for changing spin direction is

$$\left.\frac{d\sigma}{d\Omega}(\theta,\phi)\right|_{(\downarrow)} = |g(k,\theta)|^2 \tag{2.6.29}$$

The expression of the analyzing power, denoted by A_y is given by Cooper and Horowitz (2005).

$$A_y = \frac{2\,Im\left[f(k,\theta)\cdot g(k,\theta)^*\right]}{|f(k,\theta)|^2 + |g(k,\theta)|^2} \tag{2.6.30}$$

This quantity gives in particular the fractional difference in cross section for scattering to the left versus right for particles that are initially polarized transverse to the scattering plane.

The analyzing power is also a measure of the effect on scattering cross sections of changes in the polarization of the incident or target particles. (Mabiala et al., 2009)

2.7 Potential scattering

Particles involved in the scattering process have the properties such that incident particles are scattered by some potential field called potential scattering $V(r)$. The potential can be short-ranged, long-ranged, spherically symmetric and/or spin dependent. Examples of scattering potentials include nuclear potential, Coulomb potential, spin-orbital potential and optical potential.

The short-range potential is a potential which vanishes outside of a limited scattering region. In nuclear reaction, most of the potentials that arise from a certain nuclear force have similar range. They are decreasing exponentially faster than $1/r$ outside of the scattering region, and to a wave.

The Coulomb potential is a long-range potential. It is tied to the fact that particles involved in scattering process are charged. Usually this potential come, in addition to the short-range potential. For a particles with charge number Z separated by a distance r, the Coulomb potential is given by

$$U(r) = \frac{Z_i Z_t\, e^2}{r} \tag{2.7.1}$$

where Z_i and Z_t is Z number for incident and target particles and e is elementary charge. Generally, we define a Coulomb Sommerfeld parameter n given by (Satchler, 1983)

$$n = \frac{Z_i Z_t\, e^2}{\hbar v} = \frac{Z_i Z_t\, me^2}{\hbar^2 k} = \frac{Z_i Z_t\, \alpha}{(v/c)} \tag{2.7.2}$$

where $\alpha = e^2/\hbar\ c = 1/137.036$ and v is the velocity of asymptotic relative motion. To arrive at observable parameters, we replace $V(r)$ with $U(r)$ in the Schrödinger Eqn. (2.6.7). The corresponding scattering amplitude, differential cross-section and analyzing power are called Coulomb scattering amplitude $f_c(\theta)$ (which contains a Coulomb phase shift), differential cross-section $\frac{d\sigma^C(\theta)}{d\Omega}$ and analyzing power A_y^C. In nuclear experiments, both Coulomb and short-ranged potential act.

The spin-orbit potential, depends on internal spin of the scattered particle. It represents an addition to the central potential. It depends on the spin direction and therefore the spin direction is not conserved during the scattering process. The spin-orbit potential is typically represented as

$$V_{so}(\vec{r}) = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\vec{l}.\vec{s} \tag{2.7.3}$$

where l and s are orbital and spin angular momenta, m mass of scattered particle and c speed of light.

The optical potential, contains all the potential and is given by (Burke, 1977)

$$U(\vec{r}) = V_C(r) - V_r(r) - i\left(W_V(r) + W_S(r) + V_{so}(\vec{r})\right) \tag{2.7.4}$$

where $V_C(r)$ is the Coulomb potential, $V_r(r)$ is the real nuclear potential, $W_V(r)$ and $W_S(r)$ volume and surface imaginary potential and V_{so} is the spin-orbital potential.

2.8 Scattering Phase shifts

We talk about phase shift when during a scattering process, the potential can shift the phase of the wave-function. To understand the concept of the phase shifts, we consider an incident wave moving toward a potential in z-direction $A_k\, e^{ikz}$ with a shifted phase δ, the reflected wave will be scattered with the same amplitude A_k by conservation of probability explained in subsection 2.6.1, but with another shifted phase δ. Further, the scattered wave will be shifted twice $A_k e^{i(kr+2\delta)}$.

Phase shifts is most used to build a formalism of writing simply the wave-function because it gives a physical good interpretation of the wave-function at instead of writing it in term of partial wave defined in the Eqn (2.6.16). For a particular potential we can compute the phase shift by solving numerically the Schrödinger equation.

3. Analyzing power for electrically neutral spin particles

3.1 Scattering amplitude with spinless and neutral particles

Scattering amplitude can be introduced by solving the time-independent Schrödinger equation (2.6.7), ignoring what is happening within the scattering region and considering only the asymptotic case ($\vec{r} \longrightarrow \infty$). To do this is, we use the partial wave analysis method used by Rayleigh (Rayleigh, 1977) and applied for the first time the scattering of particles problem by Faxén and Holtsmark (Faxén and Holtsmark). For central potential the wave function $\psi(\vec{r})$ can be expanded in series

$$\psi(\vec{r}) = \sum_{l=0}^{\infty} \frac{B_l(k)}{r} u_l(\vec{r}) P_l(\cos\theta) \tag{3.1.1}$$

where $P_l(\cos\theta)$ are the Legendre polynomial functions described below, $u_l(\vec{r})$ is the radial wave-function and $B_l(k)$ is a coefficient depending on the quantum number k.

In spherical coordinates, wave-function is function of 3 variables (r, θ, ϕ) and can be searched for as the product of two functions, a radial function and a harmonic function:

$$\psi_n(r, \theta, \phi) = R(r)\, Y(\theta, \phi) \tag{3.1.2}$$

In spherical coordinates the Laplacian is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2}{\partial \theta^2}\right) \tag{3.1.3}$$

Substituting the Eq. (3.1.3) and the Eq. (3.1.2) into the Eq. (2.6.7), we get

$$\begin{aligned} &-\frac{\hbar^2}{2m}\left\{\frac{Y(\theta,\phi)}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R(r)}{\partial r}\right) + \frac{R(r)}{r^2 \sin\theta}\frac{\partial Y(\theta,\phi)}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{R(r)}{r^2 \sin^2\theta}\left(\frac{\partial^2 Y(\theta,\phi)}{\partial \theta^2}\right)\right\} \\ &+ V R(r) Y(\theta,\phi) = E\, V R(r) Y(\theta,\phi) \end{aligned} \tag{3.1.4}$$

Upon multiplying by $-2mr^2/\hbar^2\ Y(\theta,\phi)R(r)$ and by separation variables, we can write the equation into two equations, one dependent on position r and the other dependent on the angular position (θ, ϕ) as

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R(r)}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(V - E)R(r) = CR(r) \tag{3.1.5}$$

$$\frac{1}{Y(\theta,\phi)}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial Y(\theta,\phi)}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial \theta^2}\right] = -C \tag{3.1.6}$$

We know that the simultaneous eigenfunctions of the operators L^2 and L_z are denoted by $Y_{lm}(\theta,\phi)$ and the eigenvalue of L^2 acting on $Y_{lm}(\theta,\phi)$ is defined by $\hbar^2 l(l+1)$, such that we can write

$$L^2 Y_{lm}(\theta,\phi) = \hbar^2 l(l+1) Y_{lm}(\theta,\phi) \tag{3.1.7}$$

From the Appendix (A.3.6), the L^2 operator in spherical coordinates is given by

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \left(\frac{\partial^2}{\partial\theta^2} \right) \right] \tag{3.1.8}$$

Substituting Eq. (3.1.8) into the Eq. (3.1.7) we get

$$\left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \left(\frac{\partial^2}{\partial\theta^2} \right) \right] Y_{lm}(\theta,\phi) = l(l+1) Y_{lm}(\theta,\phi) \tag{3.1.9}$$

Therefore the constant C into the Eqn. (3.1.5) corresponds to the value $l(l+1)$ and we can rewrite the equation set as

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial R(r)}{\partial r} \right) - \frac{2mr^2}{\hbar^2} (V - E) R(r) = l(l+1) R(r) \tag{3.1.10}$$

$$\left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Y(\theta,\phi)}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 Y(\theta,\phi)}{\partial\theta^2} \right] = -l(l+1) Y(\theta,\phi) \tag{3.1.11}$$

From the Appendix (A.4.13) the solution to the angular part of equation is given by

$$Y_{lm}(\theta,\phi) = \begin{cases} (-1)^m \left[\dfrac{2l+1}{4\pi} \dfrac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(\cos\theta) e^{im\phi} \text{ for } m \geq 0 \\ \\ (-1)^m Y_{lm}^*(\theta,\phi) \text{ for } m < 0 \end{cases} \tag{3.1.12}$$

Far away from the scattering region, the potential effect vanishes so that the above radial Schrödinger equation becomes

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial R(r)}{\partial r} \right) + \frac{2mr^2 E}{\hbar^2} R(r) = l(l+1) R(r) \tag{3.1.13}$$

This can be further rewritten as

$$\frac{d}{dr} \left(r^2 \frac{dR(r)}{dr} \right) + \left((kr)^2 - l(l+1) \right) R(r) = 0 \tag{3.1.14}$$

With $\rho = kr$, after dividing b y k^2, the radial Schrödinger equation can be rewritten as

$$\rho^2 \frac{d^2R(\rho)}{d\rho^2} + 2\rho \frac{dR(\rho)}{d\rho} + \left(\rho^2 - l(l+1)\right) R(\rho) = 0 \tag{3.1.15}$$

Upon substitution

$$J(\rho) = \sqrt{\rho}\, R(\rho) \tag{3.1.16}$$

the radial Schrödinger equation becomes

$$\rho^2 \frac{d^2J(\rho)}{d\rho^2} + \rho \frac{dJ(\rho)}{d\rho} + \left(\rho^2 - (l+1/2)^2\right) J(\rho) = 0 \tag{3.1.17}$$

This is the Bessel equation of half-integer order. The solution to this equation is the Bessel function of first kind of the form

$$R(r) = j_l(kr) = \sqrt{\pi/2kr}\, J_{l+1/2}(kr) \tag{3.1.18}$$

where $j_l(kr)$ is the integer Bessel function of the first kind which can be written in terms of a combination of the Hankel functions as

$$j_l(kr) = \frac{1}{2}\left[h_l^{(1)}(kr) + h_l^{(2)}(kr)\right] \tag{3.1.19}$$

where $h_l^{(1)}(kr)$ and $h_l^{(2)}(kr)$ are respectively the spherical Hankel functions of first and second kind. For large value of r, the explicit form of the linear combination of the Hankel functions $j_l(kr)$ is given by

$$j_l(kr) \approx \frac{1}{kr} sin(kr - l\pi/2) \tag{3.1.20}$$

Then the total wave-function Eq. 3.1.2, at far away from the target is given by (Griffiths, 2005)

$$\psi(r,\theta,\phi) \approx A\left[e^{ikz} + \sum_{l,m} C_{l,m} h_l^{(1)}(kr) Y_l^m(\theta,\phi)\right] \tag{3.1.21}$$

where $C_{l,m}$ are the expansion coefficients. Using the Eq. (3.1.19) and for $m = 0$ under azimuthal symmetry we can rewrite this as (Griffiths, 2005)

$$\psi(r,\theta) \approx A\left[e^{ikz} + k\sum_{l} i^{l+1}(2l+1)a_l h_l^{(1)}(kr) P_l(\cos\theta)\right] \tag{3.1.22}$$

Since the scattering process is elastic, the quantum number k is the same for the incident and scattered particles. By identifying the Eq. (3.1.22), with the Eq. (2.6.8) we find that the scattering amplitude $f(\theta,\phi)$ is given by (Griffiths, 2005)

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1)\, a_l P_l(\cos\theta) \tag{3.1.23}$$

where $P_l(\cos\theta)$ are the Legendre polynomial functions.

As we explained in the previous chapter about the phase shift, even if we do not consider what is happening within the scattering region, the scattering potential exists and is still have effect on the scattering process and the scattered particles described by a wave-function have a shifted phase compared to the incident phase. The Hankel function being expressed in term of the phase shift, then the Eqn. (3.1.23) can be rewritten as (Griffiths, 2005)

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l \; P_l(\cos\theta) \tag{3.1.24}$$

where δ_l is the l-th phase shift.

3.2 Scattering DCS of spinless neutral particles

Getting the scattering amplitude helps to express the differential cross section $\frac{d\sigma}{d\Omega}$ in term of the scattering angle θ. From the Eq. (2.6.9) the differential cross section is

$$\frac{d\sigma}{d\Omega}(\theta) = \left| \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l \; P_l(\cos\theta) \right|^2 \tag{3.2.1}$$

where k is the quantum number linked to the energy E of the incident particles and given by the Eq. (2.6.14). m is the eigenvalue of J_z a projection of the total angular momentum J. $P_l(\cos\theta)$ explicit expressions are given in Appendix (B.1). The value of θ is the scattering direction in an experiment. δ_l is predicted or deduced for a given scattering process.

3.3 Scattering amplitude of neutral particles with spin

The asymptotic form of the wave-function describing the free motion of the particle in scattering process is given by the Eq. (2.6.8). In the case of the spin-1/2 particle, the free motion of the particle is described by wave-function associated to a given spin state defined by the quantum number $(s = 1/2, m_s)$. This kind of wave-function is called a spin-angle wave-function and scattering amplitude is a 2×2 matrix. If we consider the direction of the incident wave along the z-axis, then the asymptotic form of the wave-function is given by:(Burke, 2011)

$$\psi(\vec{r},\sigma) = \chi_{\frac{1}{2}m_s}(\sigma) e^{ikz} + \sum_{m_s'=\pm\frac{1}{2}} \chi_{\frac{1}{2}m_s'}(\sigma) M_{m_s' m_s}(\theta,\phi) \frac{e^{ikr}}{r} \tag{3.3.1}$$

where $\chi_{\frac{1}{2}m_s}(\sigma)$ is the particle spin eigenfunction, $M_{m'_s m_s}$ the scattering amplitude in matrix form for incident particle with spin m_s to the scattered particle with spin m'_s. m_s and m'_s can take as values $1/2$ or $-1/2$. σ is the vector Pauli matrix.

Let us recall the expansion of the plane wave-function e^{ikz}, Eq. (2.6.16),

$$e^{ikz} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta) \tag{3.3.2}$$

where $P_l(\cos\theta)$ are the Legendre polynomials and $j_l(kr)$ are the spherical Bessel functions of the first order which asymptotic form is given by

$$j_l(kr) \approx \frac{\sin(kr - \pi l/2)}{kr} \tag{3.3.3}$$

Let us recall also the spherical harmonic functions, solutions to the angular part of the Schrödinger equation (3.1.10), obtained in the Eqn (3.1.12) as

$$Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l(\cos\theta)e^{im\phi} \tag{3.3.4}$$

where $P_l(cos\theta)$ are independent of m. For $m=0$ the relation is

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\,Y_{l0}(\theta,\phi) \tag{3.3.5}$$

Using the fact that the particle has a spin-1/2 and is described also by the spin eigenfunction $\chi_{\frac{1}{2}m_s}(\sigma)$, the Eqn. (3.3.2) can be rewritten as

$$\chi_{\frac{1}{2}m_s}(\sigma)e^{ikz} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)\chi_{\frac{1}{2}m_s}(\sigma) \tag{3.3.6}$$

By substituting the Eqn. (3.3.5) into the Eqn. (3.3.6) the plane wave function with spin contribution becomes

$$\chi_{\frac{1}{2}m_s}(\sigma)e^{ikz} = \sum_{l=0}^{\infty}(2l+1)\sqrt{\frac{4\pi}{2l+1}}\,i^l j_l(kr)\,Y_{l0}(\theta,\phi)\chi_{\frac{1}{2}m_s}(\sigma) \tag{3.3.7}$$

which may be simplified as

$$\chi_{\frac{1}{2}m_s}(\sigma)e^{ikz} = \sum_{l=0}^{\infty}\sqrt{4\pi(2l+1)}\,i^l j_l(kr)\,Y_{l0}(\theta,\phi)\chi_{\frac{1}{2}m_s}(\sigma) \tag{3.3.8}$$

From the Eqn. (2.4.9) the spherical eigenfunction $\mathcal{Y}_{lm_l\frac{1}{2}m_s}(\theta,\phi)$ for particle with spin m_s and orbit angular momentum described by l and m_l are given by

$$\mathcal{Y}_{lm_l\frac{1}{2}m_s}(\theta,\phi,\sigma) = \sum_{m_l\, m_s} \langle lm_l\frac{1}{2}m_s|jm\rangle Y_{lm_l}(\theta,\phi)\chi_{\frac{1}{2}m_s}(\sigma) \tag{3.3.9}$$

where $\langle lm_l\frac{1}{2}m_s|jm\rangle$ is the Clebsch-Gordan coefficient defined in the Eqn. (2.4.10).

Following the properties of the Clebsch-Gordan coefficient defined in Appendix (C.0.5), one can invert the Eq. (3.3.9) to yield

$$Y_{lm_l}(\theta,\phi)\chi_{\frac{1}{2}m_s}(\sigma) = \sum_j \langle lm_l\frac{1}{2}m_s|jm\rangle \mathcal{Y}_{lsjm}(\theta,\phi,\sigma) \tag{3.3.10}$$

where j varies between $|j_1 - j_2|$ and $|j_1 + j_2|$ as defined in the Eqn. (2.4.8), $j_1 = l$ and $j_2 = s = 1/2$ in this case. In particular, for $m_l = 0$, Eq.(3.3.10) becomes

$$Y_{l0}(\theta,\phi)\chi_{\frac{1}{2}m_s}(\sigma) = \sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|} \langle l0\frac{1}{2}m_s|jm\rangle \mathcal{Y}_{l\frac{1}{2}jm}(\theta,\phi,\sigma) \tag{3.3.11}$$

where $l = 0, 1, 2, 3 \ldots\infty$ and $m = m_l + m_s = m_s$ since $m_l = 0$. Then we can modify a little bit the Eqn. (3.3.11) can be simplified to

$$Y_{l0}(\theta,\phi)\chi_{\frac{1}{2}m_s}(\sigma) = \sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|} \langle l0\frac{1}{2}m_s|jm_s\rangle \mathcal{Y}_{l\frac{1}{2}jm_s}(\theta,\phi,\sigma) \tag{3.3.12}$$

By substituting Eq. (3.3.12) into Eq. (3.3.8), we obtain

$$\chi_{\frac{1}{2}m_s}(\sigma)e^{ikz} = \sum_{l=0}^{\infty} \sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|} \sqrt{4\pi(2l+1)}\; i^l j_l(kr)\; \langle l0\frac{1}{2}m_s|jm_s\rangle \mathcal{Y}_{l\frac{1}{2}jm_s}(\theta,\phi,\sigma) \tag{3.3.13}$$

Therefore, by substituting Eq. (3.3.13) into the Eqn. (3.3.1), the spin-angle wave-function is now given by

$$\psi(\vec{r},\sigma) = \sum_{l=0}^{\infty} \sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|} \sqrt{4\pi(2l+1)}\; i^l j_l(kr)\; \langle l0\frac{1}{2}m_s|jm_s\rangle \mathcal{Y}_{l\frac{1}{2}jm_s}(\theta,\phi,\sigma) + \psi_{sc}(\vec{r},\sigma) \tag{3.3.14}$$

where $\psi_{sc}(\vec{r},\sigma)$ is the scattered part of the wave function. Asymptotically, at $r \to \infty$, $j_l(kr)$ take the form

$$j_l(kr) \approx \sin\left(kr - \frac{\pi l}{2}\right) = \frac{e^{i(kr-\frac{1}{2}l\pi)}}{2ikr} - \frac{e^{-i(kr-\frac{1}{2}l\pi)}}{2ikr} \tag{3.3.15}$$

In spin-space Eq. (3.1.1) can be written as

$$\psi(\vec{r},\sigma) = \sum_{l=0}^{\infty} \sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|} \frac{B_{lj}(k)}{r} u_{lj}(\vec{r}) \mathcal{Y}_{l\frac{1}{2}jm_s}(\theta,\phi,\sigma) \tag{3.3.16}$$

where $u_{lj}(r)$ are the radial functions and for the asymptotic form of the wave function to be verified, the radial functions are given by

$$u_{lj}(r) \approx \sin\left(kr - \frac{\pi l}{2} - \delta_{lj}(k)\right) = \frac{e^{i(kr-\frac{1}{2}l\pi+\delta_{lj})}}{2ikr} - \frac{e^{-i(kr-\frac{1}{2}l\pi+\delta_{lj})}}{2ikr} \tag{3.3.17}$$

where $\delta_{lj}(k)$ is the phase shift explained in section 2.8.

To determine the coefficient $B_{lj}(k)$ we need to demand that the two incoming wave functions in Eq. (3.3.13) and Eq. (3.3.16) are equal. We get

$$\frac{B_l(k)}{2ir} e^{-i(kr-\frac{1}{2}l\pi+\delta_l)} = \frac{1}{2ikr}\sqrt{4\pi(2l+1)}\; i^l\; \langle l0\frac{1}{2}m_s|jm_s\rangle e^{-i(kr-\frac{1}{2}l\pi)} \tag{3.3.18}$$

Then, we can extract the constant $B_l(k)$ and get

$$B_l(k) = \sqrt{4\pi(2l+1)}\; i^l \frac{e^{i\delta_l}}{k} \langle l0\frac{1}{2}m_s|jm_s\rangle \tag{3.3.19}$$

Therefore the Eqn. (3.3.16) becomes

$$\psi(\vec{r},\sigma) = \sum_{l=0}^{\infty} \sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|} \sqrt{4\pi(2l+1)}\; i^l \frac{e^{\delta_{jl}(k)}}{kr} \sin(kr - \pi l/2 - \delta_{lj}(k)) \langle l0\frac{1}{2}m_s|jm_s\rangle \mathcal{Y}_{l\frac{1}{2}jm_s}(\theta,\phi,\sigma) \tag{3.3.20}$$

The matrix scattering amplitude M_{m_s',m_s} which we are looking for, is in the second term of the Eq. (3.3.1). To get this second term $\psi_{sc}(\vec{r},\sigma)$, we subtract the first term on the r.h.s of Eq. (3.3.14) from the r.h.s of Eq. (3.3.20). The asymptotic form of the scattered wave function then emerges as

$$\begin{aligned}\psi_{sc}(\vec{r},\sigma) = &\sum_{l=0}^{\infty} \sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|} \sqrt{4\pi(2l+1)}\; i^l \frac{e^{\delta_{jl}(k)}}{kr} \sin(kr - \pi l/2 - \delta_{lj}(k)) \langle l0\frac{1}{2}m_s|jm_s\rangle \mathcal{Y}_{l\frac{1}{2}jm_s}(\theta,\phi,\sigma) - \\ &\sum_{l=0}^{\infty} \sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|} \sqrt{4\pi(2l+1)}\; i^l \frac{\sin(kr-\pi l/2)}{kr} \langle l0\frac{1}{2}m_s|jm_s\rangle \mathcal{Y}_{l\frac{1}{2}jm_s}(\theta,\phi,\sigma)\end{aligned} \tag{3.3.21}$$

This gives simply

$$\psi_{sc}(\vec{r},\sigma) = \frac{1}{kr}\sum_{l=0}^{\infty}\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sqrt{4\pi(2l+1)}\, i^l \left(e^{\delta_{jl}(k)}\sin(kr-\pi l/2-\delta_{lj}(k)) - \sin(kr-\pi l/2)\right)$$
$$\langle l0\frac{1}{2}m_s|jm_s\rangle \mathcal{Y}_{l\frac{1}{2}jm_s}(\theta,\phi,\sigma) \tag{3.3.22}$$

Using the exponential form of sin function, i.e. $\sin\ a = \frac{e^{ia}-e^{-ia}}{2i}$, we get

$$\psi_{sc}(\vec{r},\sigma) = \frac{1}{2ikr}\sum_{l=0}^{\infty}\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sqrt{4\pi(2l+1)}\, i^l e^{-il\pi/2}\left(e^{i2\delta_{lj}(k)}-1\right)e^{ikr}\langle l0\frac{1}{2}m_s|jm_s\rangle \mathcal{Y}_{l\frac{1}{2}jm_s}(\theta,\phi,\sigma) \tag{3.3.23}$$

Since $i^l e^{-il\pi/2} = e^{l\ln i}e^{-il\pi/2} = e^{l\ln e^{i\pi/2}}e^{-il\pi/2} = e^{il\pi/2}e^{-il\pi/2} = 1$, then we can simplify the above into

$$\psi_{sc}(\vec{r},\sigma) = \frac{1}{2ik}\sum_{l=0}^{\infty}\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sqrt{4\pi(2l+1)}\left(e^{i2\delta_{lj}(k)}-1\right)\langle l0\frac{1}{2}m_s|jm_s\rangle \mathcal{Y}_{l\frac{1}{2}jm_s}(\theta,\phi,\sigma)\frac{e^{ikr}}{r} \tag{3.3.24}$$

After we substitute Eq. (3.3.9) into Eq. (3.3.24), we arrive at the explicit form of the second term of Eq. (3.3.1)

$$\psi_{sc}(\vec{r},\sigma) = \frac{1}{2ik}\sum_{m_s'=\pm\frac{1}{2}}\sum_{l=0}^{\infty}\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sqrt{4\pi(2l+1)}\left(e^{i2\delta_{lj}(k)}-1\right)\langle l0\frac{1}{2}m_s|jm_s\rangle\langle lm_l\frac{1}{2}m_s|jm\rangle$$
$$\times Y_{lm_l}(\theta,\phi)\chi_{\frac{1}{2}m_s}(\sigma)\frac{e^{ikr}}{r} \tag{3.3.25}$$

By identification, from Eq.(3.3.1), the matrix scattering amplitude is given by

$$M_{m_s'm_s}(\theta,\phi) = \frac{1}{2ik}\sum_{l=0}^{\infty}\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sqrt{4\pi(2l+1)}\left(e^{i2\delta_{lj}(k)}-1\right)\langle l0\frac{1}{2}m_s|jm_s\rangle\langle lm_l\frac{1}{2}m_s|jm\rangle Y_{lm_l}(\theta,\phi) \tag{3.3.26}$$

where $m = m_s$ and $m = m_l + m_s'$, implying $m_l = m_s - m_s'$. Therefore the matrix scattering amplitude can be expressed as

$$M_{m_s'm_s}(\theta,\phi) = \frac{1}{2ik}\sum_{l=0}^{\infty}\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sqrt{4\pi(2l+1)}\left(e^{i2\delta_{lj}(k)}-1\right)\langle l0\frac{1}{2}m_s|jm_s\rangle\langle lm_s-m_s'\frac{1}{2}m_s'|jm_s\rangle$$
$$Y_{lm_s-m_s'}(\theta,\phi) \tag{3.3.27}$$

where $\langle l0\frac{1}{2}m_s|jm_s\rangle$ and $\langle lm_s - m'_s\frac{1}{2}m'_s|jm_s\rangle$ are the Clebsch-Gordan coefficient and $Y_{lm_s-m'_s}(\theta,\phi)$ are the spherical harmonics of which explicit form is given in the Appendix (B.3). This allows to express the matrix elements of the matrix scattering amplitude obtained in the Eqn. (**??**) in terms of phase shifts.

To get the direct or the spin-non flip $f(\theta,\phi)$ and the spin-flip $g(\theta,\phi)$ scattering amplitude given in the Eqn. (**??**), we consider the cases of $j = l \pm 1/2$, $m = m_s = \pm 1/2$, $m = 1/2$ and $m'_s = -1/2$, since $m_l = 0$. The explicit expressions, for Clebsch-Gordan coefficient are given in the Appendix (C.1) and they yield initially

$$\langle l0\frac{1}{2}m_s|jm_s\rangle = \sqrt{\frac{l+\frac{1}{2}+\frac{1}{2}}{2l+1}} = \sqrt{\frac{l+1}{2l+1}} \qquad \texttt{for} \qquad j = l+\frac{1}{2},\ m = m_s = \frac{1}{2} \tag{3.3.28}$$

and

$$\langle l0\frac{1}{2}m_s|jm_s\rangle = \sqrt{\frac{l-\frac{1}{2}+\frac{1}{2}}{2l+1}} = \sqrt{\frac{l}{2l+1}} \qquad \texttt{for} \qquad j = l-\frac{1}{2},\ m = m_s = -\frac{1}{2} \tag{3.3.29}$$

Similarly, in the context of the spherical harmonics, the explicit form of the Clebsch-Gordan coefficient yields

$$\langle lm_s - m'_s\frac{1}{2}m'_s|jm_s\rangle = \sqrt{\frac{l+\frac{1}{2}+\frac{1}{2}}{2l+1}} = \sqrt{\frac{l+1}{2l+1}} \quad \texttt{for} \quad j = l+\frac{1}{2},\ m_s = m'_s = \frac{1}{2} \tag{3.3.30}$$

Therefore one set of spherical harmonics of interest in Eq. (3.1.12) becomes

$$Y_{l0}(\theta) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta) \quad \texttt{for} \quad j = l+\frac{1}{2},\ m_s - m'_s = 0 \tag{3.3.31}$$

For the Clebsch-Gordan coefficients, we further have

$$\langle lm_s - m'_s\frac{1}{2}m'_s|jm_s\rangle = \sqrt{\frac{l-\frac{1}{2}+\frac{1}{2}}{2l+1}} = \sqrt{\frac{l}{2l+1}} \quad \texttt{for} \quad j = l-\frac{1}{2},\ m_s = -\frac{1}{2},\ m'_s = -\frac{1}{2} \tag{3.3.32}$$

with this, the explicit form of another set of spherical harmonics becomes

$$Y_{l1}(\theta) = -\sqrt{\frac{2l+1}{4\pi\, l(l+1)}}\, P^1_l(\cos\theta) \quad \texttt{for} \quad j = l-\frac{1}{2},\ m_s - m'_s = 1 \tag{3.3.33}$$

Applying Eq. (3.3.28) - (3.3.33) to Eq. (3.3.27) we get

$$f(k,\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}\left[(l+1)\left(e^{2i\delta_{l,l+\frac{1}{2}}(k)}-1\right)+l\left(e^{2i\delta_{l,l-\frac{1}{2}}(k)}-1\right)\right]P_l(\cos\theta) \tag{3.3.34}$$

and

$$g(k,\theta) = \frac{1}{2ik}\sum_{l=1}^{\infty}\left(e^{2i\delta_{l,l+\frac{1}{2}}(k)}-e^{2i\delta_{l,l-\frac{1}{2}}(k)}\right)P_l^{(1)}(\cos\theta) \tag{3.3.35}$$

where $f(k,\theta)$ and $g(k,\theta)$ are respectively direct scattering amplitude and spin-flip scattering amplitude. Here $P_l(\cos\theta)$ are the Legendre polynomial functions with explicit expressions provided in Appendix (B.1).

3.4 Scattering DCS for neutral particles with spin

As explained in the previous chapter, from Eq. (2.6.9), the differential cross section for particles moving from an incident state with a quantum number k and spin state m_s to a scattered state with the same quantum number k (for elastic scattering) and a spin state m_s' is given by

$$\frac{d\sigma}{d\Omega}(\theta,\phi) = \sum_{m_s,m_s'}\frac{1}{2s+1}\left|M_{m_s'm_s}(\theta,\phi)\right|^2 \tag{3.4.1}$$

Using the matrix scattering amplitude obtained in Eq. (3.3.27) we get

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\theta,\phi) = \sum_{m_s,m_s'}\frac{1}{2s+1}\Bigg|&\frac{1}{2ik}\sum_{l=0}^{\infty}\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sqrt{4\pi(2l+1)}\left(e^{i2\delta_{lj}(k)}-1\right)\langle l0\frac{1}{2}m_s|jm_s\rangle\\ &\langle lm_s-m_s'\frac{1}{2}m_s'|jm_s\rangle Y_{lm_s-m_s'}(\theta,\phi)\Bigg|^2\end{aligned} \tag{3.4.2}$$

This can be expanded like

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\theta,\phi) =&\frac{1}{2s+1}\Bigg[\left(\frac{1}{2ik}\right)\left(\frac{1}{2ik}\right)^*\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sqrt{4\pi(2l+1)}\sqrt{4\pi(2l'+1)}\\ &\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sum_{j=|l'-\frac{1}{2}|}^{|l'+\frac{1}{2}|}\left(e^{i2\delta_{lj}(k)}-1\right)\left(e^{i2\delta_{lj}(k)}-1\right)^*\sum_{m_s,m_s'}\langle l0\frac{1}{2}m_s|jm_s\rangle\langle lm_s-m_s'\frac{1}{2}m_s'|jm_s\rangle\\ &\langle l'0\frac{1}{2}m_s|jm_s\rangle\langle l'm_s-m_s'\frac{1}{2}m_s'|jm_s\rangle\; Y_{lm_s-m_s'}(\theta,\phi)\; Y^*_{lm_s-m_s'}(\theta,\phi)\Bigg]\end{aligned} \tag{3.4.3}$$

For an incident particle with initial spin state up: $m_s = \frac{1}{2}$ and scattered particle with final spin state up $m'_s = \frac{1}{2}$, the differential cross section is given by

$$\frac{d\sigma}{d\Omega}(\theta,\phi)\bigg|_{\uparrow\uparrow} = \Bigg[\left(\frac{1}{2ik}\right)\left(\frac{1}{2ik}\right)^{*}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sqrt{4\pi(2l+1)}\sqrt{4\pi(2l'+1)}$$
$$\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sum_{j=|l'-\frac{1}{2}|}^{|l'+\frac{1}{2}|}\left(e^{i2\delta_{lj}(k)}-1\right)\left(e^{i2\delta_{lj}(k)}-1\right)^{*}\langle l0\frac{1}{2}\frac{1}{2}|j\frac{1}{2}\rangle\langle l0\frac{1}{2}\frac{1}{2}|j\frac{1}{2}\rangle \tag{3.4.4}$$
$$\langle l'0\frac{1}{2}\frac{1}{2}|j\frac{1}{2}\rangle\langle l'0\frac{1}{2}\frac{1}{2}|j\frac{1}{2}\rangle\ Y_{l0}(\theta,\phi)\ Y^{*}_{l0}(\theta,\phi)\Bigg]$$

For an incident particle with initial spin state up: $m_s = \frac{1}{2}$ and scattered particle with final spin state down, $m'_s = -\frac{1}{2}$, the differential cross section is given by

$$\frac{d\sigma}{d\Omega}(\theta,\phi)\bigg|_{\uparrow\downarrow} = \Bigg[\left(\frac{1}{2ik}\right)\left(\frac{1}{2ik}\right)^{*}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sqrt{4\pi(2l+1)}\sqrt{4\pi(2l'+1)}$$
$$\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sum_{j=|l'-\frac{1}{2}|}^{|l'+\frac{1}{2}|}\left(e^{i2\delta_{lj}(k)}-1\right)\left(e^{i2\delta_{lj}(k)}-1\right)^{*}\langle l0\frac{1}{2}\frac{1}{2}|j\frac{1}{2}\rangle\langle l1\frac{1}{2}-\frac{1}{2}|j\frac{1}{2}\rangle \tag{3.4.5}$$
$$\langle l'0\frac{1}{2}\frac{1}{2}|j\frac{1}{2}\rangle\langle l'1\frac{1}{2}-\frac{1}{2}|j\frac{1}{2}\rangle\ Y_{l1}(\theta,\phi)\ Y^{*}_{l1}(\theta,\phi)\Bigg]$$

where k is the quantum number linked to the energy E of the incident particles (which is the same for the elastic scattering) and given by the Eqn. (2.6.14). Under symmetry and consider the scattering process to be in the plane, ϕ equal 0.

4. Analyzing power for charged particles

4.1 Coulomb scattering process

4.1.1 Coulomb wave-function and scattering amplitude.

In Chapter (3) we have seen that the wave-function describing the motion of particles from the initial state to the final state is function of three variables (r, θ, ϕ), obtained from the both radial and angular Schrödinger differential equation in Eq. (3.1.10). The scattering potential does not affect the angular part of the Schrödinger differential equation, but can modify the radial differential equation, which contains the potential $V(r)$.

The task in this section is to determine the radial function defined in the Eqn.(3.1.19) by including the Coulomb potential since particles involved in the scattering process are charged. To do this, we recall the Coulomb potential defined in the Eqn. (2.7.1) and substitute it into the Eqn. (2.6.7) and find the asymptotic form of the Coulomb wave-function, in which we can extract the Coulomb scattering amplitude needed.

The time independent Schrödinger equation is given by

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + (V_c(r) - E)\right]\psi_c(r) = 0 \tag{4.1.1}$$

where $V_c(r)$ is the Coulomb potential, and m and E the mass and energy of the particle. $V_c(r)$ is given by

$$V_c(r) = \frac{Z_i Z_t\, e^2}{r} \tag{4.1.2}$$

Substituting this into the time independent Schrödinger equation gives

$$\left[\nabla^2 + k^2 - \frac{2\eta k}{r}\right]\psi_c(r) = 0 \tag{4.1.3}$$

where η is the Sommerfeld parameter given by

$$\eta = \frac{Z_i Z_t\, m e^2}{\hbar^2 k} \tag{4.1.4}$$

Let us consider the parabolic coordinates of the equation (4.1.3). To solve this, we set

$$\lambda = r - z;\ \zeta = r + z;\ \text{ and } \phi = \tan^{-1} y/x \tag{4.1.5}$$

where (x, y, z) are Cartesian coordinates and (λ, ζ, ϕ) are parabolic coordinates. In parabolic coordinates, the Laplacian ∇^2 is given by

$$\nabla^2 = \frac{4}{r-z}\left[\frac{\partial}{d\lambda}\left(\lambda\frac{\partial}{d\lambda}\right) + \frac{\partial}{d\zeta}\left(\zeta\frac{\partial}{d\zeta}\right)\right] + \frac{1}{\lambda\,\zeta}\frac{\partial^2}{d\phi^2} \tag{4.1.6}$$

Implementing this into the differential equation (4.1.3) gives

$$\left(-\frac{4}{\lambda+\zeta}\left[\frac{\partial}{d\lambda}\left(\lambda\frac{\partial}{d\lambda}\right) + \frac{\partial}{d\zeta}\left(\zeta\frac{\partial}{d\zeta}\right)\right] + \frac{1}{\lambda\,\zeta}\frac{\partial^2}{d\phi^2} + \frac{\eta\,k}{\lambda+\zeta}\right)\psi_c = k^2\psi_c \tag{4.1.7}$$

Let us set

$$\psi_c(r) = e^{ik\lambda}F(\lambda) = e^{ik(r-z)}F(\lambda) \tag{4.1.8}$$

so that the Eq. (4.1.7) can be reduced as

$$\lambda\frac{d^2F(\lambda)}{d\lambda^2} + (1-ik\lambda)\frac{dF(\lambda)}{d\lambda} - i\eta F(\lambda) = 0 \tag{4.1.9}$$

This is the Coulomb differential equation. The standard form of this equation is the **Confluent Hypergeometric Equation** called **the Kummer's differential equation**. The simple form of the Kummer's differential equation is given by

$$z\frac{d^2y}{dz^2} + (b-z)\frac{dy}{dz} - a\,y = 0 \tag{4.1.10}$$

The solution to the Kummer's equation is a Kummer's function of the first kind called also the Confluent Hyper-geometric function given by

$$y(z) =_1 F_1(a;b;z) \tag{4.1.11}$$

The contour integral form of the confluent hyper-geometric function, defined in the Eq. (4.1.11), is given by

$${}_1F_1(a;b;z) = \frac{\Gamma(b)}{2\pi\ i}\oint_\gamma e^t\ t^{a-b}(t-z)^{-a}dt \tag{4.1.12}$$

By definition of the Gamma function, this can be rewritten as

$${}_1F_1(a;b;z) = \frac{(b-1)!}{2\pi\ i}(-z)^{-a}\oint_\gamma e^t\ t^{a-b}\left(1-\frac{t}{z}\right)^{-a} dt \tag{4.1.13}$$

where γ is a path which encloses certain singular points (example of the origin point of scattering) in the anti-clockwise direction. If we consider two singular points $t=0$ and $t=z$, the closed part can be decomposed in two paths so that Eq.(4.1.13) ca be rewritten as

$$
{}_1F_1(a;b;z) = \frac{(b-1)!}{2\pi\ i}(-z)^{-a}\oint_{\gamma_1} e^t\ t^{a-b}\left(1-\frac{t}{z}\right)^{-a} dt + \frac{(b-1)!}{2\pi\ i}\ z^{-a}\oint_{\gamma_2} e^t\ t^{-a}\left(1-\frac{t}{z}\right)^{a-b} dt \tag{4.1.14}
$$

Using the definition of the integral form of the Gamma function and using the expansion form of the hyper-geometric function respectively defined below

$$
\frac{1}{2\pi\ i}\oint_{\gamma} e^t\ t^{t-x} dt = \frac{1}{\Gamma(x)} \tag{4.1.15}
$$

and

$$
{}_1F_1(a;b;z) = \sum_{n=0}^{\infty}\frac{(a)_n\ z^n}{(b)_n\ n!} \tag{4.1.16}
$$

where $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$. Then, the Eq.(4.1.14) can be rewritten as

$$
{}_1F_1(a;b;z) = \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}G(a,a-b+1,-z) + \frac{\Gamma(b)}{\Gamma(a)}e^z\ z^{a-b}\ G(-a,b-a,z) \tag{4.1.17}
$$

where $G(a,b,z)$ is a convergence series defined by

$$
G(a,b,z) = 1 + \frac{ab}{z\ 1!} + \frac{a(a+1)\ b(b+1)}{z^2\ 2!} + \cdots \tag{4.1.18}
$$

Using the the fact we can write $x^y = e^{ylnx}$, the above solution can be rewritten as

$$
{}_1F_1(a;b;z) = \frac{\Gamma(b)}{\Gamma(b-a)}e^{-a\ln(-z)}G(a,a-b+1,-z) + \frac{\Gamma(b)}{\Gamma(a)}e^{z+(a-b)\ln(z)}\ G(-a,b-a,z) \tag{4.1.19}
$$

By comparing the solution (4.1.11), our solution to the Eq. (4.1.9) is given by

$$
F(\lambda) =_1 F_1(-in,1,ik\lambda) \tag{4.1.20}
$$

For large λ, the asymptotic behaviour of that solution is

$$
\begin{aligned}
{}_1F_1(-in,1,ik\lambda) =& \frac{\Gamma(1)}{\Gamma(1+i\eta)}e^{-(i\eta)\ln(-ik\lambda)}\left(1-\frac{\eta^2}{ik\lambda}+\cdots\right) + \frac{\Gamma(1)}{\Gamma(-i\eta)}e^{ik\lambda-(i\eta+1)\ln(ik\lambda)} \\
&\left(1-\frac{(1+\eta)^2}{ik\lambda}+\cdots\right)
\end{aligned} \tag{4.1.21}
$$

We have made $\Gamma(1) = 1$, and at large distances $(1 - \frac{\eta^2}{ik\lambda} + \cdots) \approx 1$ and $(1 - \frac{(1+\eta)^2}{ik\lambda} + \cdots) \approx 1$, so that we can write the solution as

$$ {}_1F_1(-in, 1, ik\lambda) \simeq \frac{1}{\Gamma(1+i\eta)} \left[e^{-(i\eta)\ln(-ik\lambda)} + \frac{\Gamma(1+i\eta)}{\Gamma(-i\eta)} e^{ik\lambda-(i\eta+1)\ln(ik\lambda)} \right] \tag{4.1.22} $$

Using the conditions (4.1.5) and (4.1.8), the wave-function at large distances acquires the form

$$ \psi_c(r) \simeq \frac{e^{ikz}}{\Gamma(1+i\eta)} \left[e^{-(i\eta)\ln(-ik(r-z))} + \frac{\Gamma(1+i\eta)}{\Gamma(-i\eta)} e^{ik(r-z)-(i\eta+1)\ln(ik(r-z))} \right] \tag{4.1.23} $$

This can be simplified as

$$ \psi_c(r) \simeq \frac{e^{\eta\pi/2}}{\Gamma(1+i\eta)} \left[e^{i(kz+i\eta\ln(k(r-z))} + \frac{\Gamma(1+i\eta)}{\Gamma(-i\eta)} e^{i(kr-\eta\ln(kr(1-\cos\theta))-\pi/2)} e^{-\ln(k\lambda)} \right] \tag{4.1.24} $$

with $e^{-\ln(k\lambda)} = \frac{1}{k\lambda} = \frac{1}{kr(1-\cos\theta)}$ and $1 - \cos\theta = 2\sin^2(\theta/2)$, the solution becomes

$$ \psi_c(r) \simeq \frac{e^{\eta\pi/2}}{\Gamma(1+i\eta)} \left[e^{i(kz+i\eta\ln(k(r-z))} + \frac{\Gamma(1+i\eta)}{\Gamma(-i\eta)} \frac{e^{-i\eta\ln(\sin^2(\theta/2))}}{2k\sin^2(\theta/2)} \frac{e^{i(kr-\eta\ln(2kr)-\pi/2)}}{r} \right] \tag{4.1.25} $$

Using the property of the Gamma function $\Gamma(z) = \frac{\Gamma(1+z)}{z}$ and knowing that $i = e^{-i\pi/2}$, we can write

$$ \frac{\Gamma(1+i\eta)}{\Gamma(-i\eta)} = -\eta\, e^{2i\sigma_0}\, e^{-i\pi/2} \tag{4.1.26} $$

where $\sigma_0 = \arg\Gamma(1+i\eta)$ is the Coulomb phase shift and $e^{2i\sigma_0} = \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}$. This allows to rewrite asymptotic form of the solution as

$$ \psi_c(r) \simeq \frac{e^{\eta\pi/2}}{\Gamma(1+i\eta)} \left[e^{i(kz+i\eta\ln(k(r-z))} - \frac{\eta\, e^{-i\eta\ln(\sin^2(\theta/2))+2\sigma_0}}{2k\sin^2(\theta/2)} \frac{e^{i(kr-\eta\ln(2kr))}}{r} \right] \tag{4.1.27} $$

Comparing this solution to the asymptotic solution (4.1.3), we can deduce that the Coulomb scattering amplitude as

$$ F_c(\theta) = -\frac{\eta\, e^{-i\eta\ln(\sin^2(\theta/2))+2\sigma_0}}{2k\sin^2(\theta/2)} \tag{4.1.28} $$

with η the Sommerfeld parameter, σ_0 the Coulomb phase-shift, θ the scattering angle and k the quantum number for the incident particle. Rather than by (4.1.3), the asymptotic behaviour of the Coulomb wave is actually given through by

$$\psi(r) \approx e^{i(kz+\eta\ln(2k(r-z)))} + F_c(\theta)\frac{1}{r}e^{i(kr-\eta\ln(2kr))} \tag{4.1.29}$$

The extra phase in the exponentials is associated with the fact that the Coulomb potential decreases relatively slowly with distance.

From the equation (3.1.1), we can also expand the Coulomb wave function (4.1.27) in partial waves as

$$\psi_c(r) = \sum_{l=0}^{\infty} \frac{B_l^c(k)}{r} u_l^c(r) P_l(\cos\theta) \tag{4.1.30}$$

where $u_l^c(r)$ is a solution to the radial Schrödinger equation in spherical coordinates

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) - \frac{2mr^2}{\hbar^2}\left(\frac{Z_iZ_t\,e^2}{r} - E\right)R(r) = l(l+1)R(r) \tag{4.1.31}$$

Using the Sommerfeld parameter η and the quantum number $k = f(E)$, the radial differential equation is reduced to

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + \left((kr)^2 - 2\eta kr - l(l+1)\right)R(r) = 0 \tag{4.1.32}$$

Upon introducing $\rho = kr$ and after dividing by k^2r^2, the radial Schrödinger equation can be rewritten as

$$\frac{d^2R(\rho)}{d\rho^2} + \frac{2}{\rho}\frac{dR(\rho)}{d\rho} + \left(1 - \frac{2\eta}{\rho} - \frac{l(l+1)}{\rho^2}\right)R(\rho) = 0 \tag{4.1.33}$$

This is the form of the Coulomb wave equation.Upon introducing

$$R(\rho) = \rho^l\, e^{i\rho}\, U(\rho) \tag{4.1.34}$$

the Coulomb wave equation takes the form

$$\rho\frac{d^2U(\rho)}{d\rho^2} + 2(l+1+i\rho)\frac{dU(\rho)}{d\rho} + 2\left[(l+1)i - \eta\right]U(\rho) = 0 \tag{4.1.35}$$

Upon variable change $\rho = \frac{1}{2}iz$, the equation takes the form

$$z\frac{d^2U(z)}{dz^2} + (2l+2-z)\frac{dU(z)}{dz} - (i\eta+l+1)U(z) = 0 \tag{4.1.36}$$

Comparing the simple form of the Kummer's equation and the our Coulomb differential equation defined in the Eqn. (4.1.10), our solution to this equation is

$$U(z) =_1 F_1(i\eta + l + 1; 2l + 2; z) \tag{4.1.37}$$

where ${}_1F_1(i\eta + l + 1; 2l + 2; z)$ is given by

$$\begin{aligned} {}_1F_1(i\eta + l + 1; 2l + 2; z) =& \frac{\Gamma(2l+2)}{\Gamma(l+1-i\eta)} e^{(-i\eta-l-1)\ln(-z)}\, G(i\eta + l + 1, i\eta - l, -z) \\ &+ \frac{\Gamma(2l+2)}{\Gamma(l+1+i\eta)} e^{z+(i\eta-l-1)\ln(z)}\, G(-i\eta - l - 1, l + 1 - i\eta, z) \end{aligned} \tag{4.1.38}$$

Upon introducing $\rho = \frac{1}{2} iz$, the solution $U(\rho)$ cf. (4.1.37), takes the form

$$\begin{aligned} U(\rho) = \frac{\Gamma(2l+2)}{\Gamma(l+1-i\eta)} e^{(-i\eta-l-1)\ln(2i\rho)}\, G(i\eta + l + 1, i\eta - l, 2i\rho) \frac{\Gamma(2l+2)}{\Gamma(l+1+i\eta)} \\ + e^{-2i\rho+(i\eta-l-1)\ln(-2i\rho)}\, G(-i\eta - l - 1, l + 1 - i\eta, -2i\rho) \end{aligned} \tag{4.1.39}$$

As we have assumed $R(\rho) = \rho^l\, e^{i\rho}\, U(\rho)$, with $\rho = kr$, the solution in term of r becomes

$$\begin{aligned} R(r) =& \frac{\pi\eta}{2^{l+2}} \frac{\Gamma(2l+2)}{kr} \left[\frac{e^{i(kr-l\pi/2-\ln(2ikr))} e^{-i\pi/2}}{\Gamma(l+1-i\eta)}\, G(i\eta + l + 1, i\eta - l, 2ikr) \right. \\ &\left. + \frac{e^{-i(kr-l\pi/2-\ln(2ikr))} e^{i\pi/2}}{\Gamma(l+1+i\eta)}\, G(-i\eta - l - 1, l + 1 - i\eta, -2ikr) \right] \end{aligned} \tag{4.1.40}$$

From the definition of the function $G(a, b, z)$ in the Eqn.(4.1.18), we can notice that $G(i\eta + l + 1, i\eta - l, 2ikr)$ and $G(-i\eta - l - 1, l + 1 - i\eta, -2ikr)$ are proportional to $1/kr$. Practically, during the scattering experiments, the value of kr is very large (Newton, 1982) and the norm of all the term with $1/kr$ is much less than 1. Since the first term is 1, therefore we can approximate $G(i\eta + l + 1, i\eta - l, 2ikr) \approx 1$ and $G(-i\eta - l - 1, l + 1 - i\eta, -2ikr) \approx 1$. In this case, the solution becomes

$$R(r) \simeq \frac{\pi\eta}{2^{l+2}} \frac{\Gamma(2l+2)}{kr} \left[\frac{e^{i(kr-l\pi/2-\ln(2ikr))} e^{-i\pi/2}}{\Gamma(l+1-i\eta)} + \frac{e^{-i(kr-l\pi/2-\ln(2ikr))} e^{i\pi/2}}{\Gamma(l+1+i\eta)} \right] \tag{4.1.41}$$

Multiplying and dividing by $\Gamma(l + 1 + i\eta)$ we get

$$R(r) \simeq \frac{\pi\eta}{2^{l+2}} \frac{\Gamma(2l+2)}{kr\Gamma(l+1+i\eta)} \left[\frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)} e^{i(kr-l\pi/2-\ln(2ikr))} e^{-i\pi/2} + e^{-i(kr-l\pi/2-\ln(2ikr))} e^{i\pi/2} \right] \tag{4.1.42}$$

With

$$\frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)} = e^{2i\sigma_l} \tag{4.1.43}$$

where $\sigma_l = \arg\Gamma(l+1+i\eta)$, the asymptotic solution becomes

$$R(r) \simeq \frac{\pi\eta}{2^{l+2}}\frac{\Gamma(2l+2)}{kr\Gamma(l+1+i\eta)}\left[e^{2i\sigma_l}e^{i(kr-l\pi/2-\ln(2ikr))}e^{-i\pi/2} + e^{-i(kr-l\pi/2-\ln(2ikr))}e^{i\pi/2}\right] \tag{4.1.44}$$

This can be further rewritten as

$$R(r) \simeq \frac{i\pi\eta}{2^{l+2}}\frac{\Gamma(2l+2)}{\Gamma(l+1+i\eta)}\frac{e^{i\sigma_l}}{kr}\left[e^{i(kr-l\pi/2-\ln(2ikr)+\sigma_l)} - e^{-i(kr-l\pi/2-\ln(2ikr)+\sigma_l)}\right] \tag{4.1.45}$$

Using the fact that $\sin(a) = \left(e^{ia} - e^{-ia}\right)/2i$, we can simplify the solution as

$$R(r) \simeq \frac{\pi\eta}{2^{l+1}}\frac{\Gamma(2l+2)}{\Gamma(l+1+i\eta)}\frac{e^{i\sigma_l}}{kr}\sin\left((kr - l\pi/2 - \ln(2ikr) + \sigma_l\right) \tag{4.1.46}$$

This can be written also as

$$R(r) \approx c_l\frac{e^{i\sigma_l}}{kr}\sin\left((kr - l\pi/2 - \ln(2ikr) + \sigma_l\right) \tag{4.1.47}$$

where $c_l = \frac{\pi\eta}{2^{l+1}}\frac{\Gamma(2l+2)}{\Gamma(l+1+i\eta)}$ is a constant. Usually, the solution to the Coulomb equation (4.1.31) has two solutions, the regular solution $F_l(r)$ and the irregular solution $G_l(r)$ with the asymptotic behaviour given below (MOTT and MASSEY, 1971)

$$F_l(r) \approx c_l\frac{e^{i\sigma_l}}{kr}\sin\left((kr - l\pi/2 - \ln(2ikr) + \sigma_l\right) \tag{4.1.48}$$

and

$$G_l(r) \approx b_l\frac{e^{i\sigma_l}}{kr}\cos\left((kr - l\pi/2 - \ln(2ikr) + \sigma_l\right) \tag{4.1.49}$$

With those we can write the general solution solution asymptotically as

$$R_l(r) \approx \frac{e^{i\sigma_l}}{kr}\sin\left(kr - \frac{1}{2}l\pi - \eta\ln(2kr) + \sigma_l + \delta_l\right) \tag{4.1.50}$$

with η the Sommerfeld parameter, Above c_l and b_l are some constants, δ_l is the additional phase shift due to the short-range potential and σ_l is the Coulomb phase shift.

With the above, the partial expansion of the Coulomb wave function becomes

$$\psi_c(r) = \frac{1}{kr}\sum_{l=0}^{\infty}(2l+1)\; i^l e^{i\sigma_l} F_l(\eta, kr)\; P_l(\cos\theta) \tag{4.1.51}$$

where $F_l(\eta, kr)$ is a regular solution of the confluent hyper-geometric equation obtained in Eq. (4.1.48). Using the Euler's formula, we can express $F_l(\eta, kr)$ in terms of the incoming and outgoing wave as

$$F_l(\eta, kr) \approx \frac{e^{i\sigma_l}}{2ikr} e^{i(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l)} - \frac{e^{i\sigma_l}}{2ikr} e^{-i(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l)} \tag{4.1.52}$$

Correspondingly, the expansion into partial wave becomes, asymptotically

$$\psi_c(r) \simeq \sum_{l=0}^{\infty}(2l+1)\; i^l \frac{e^{i\sigma_l}}{2ikr}\left(e^{i(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l)} - e^{-i(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l)}\right)\; P_l(\cos\theta) \tag{4.1.53}$$

4.1.2 Scattering amplitude for short-range interaction.

In practice, during the scattering process it is common for the potential to be combination of the Coulomb potential and potential due to the effect of the finite size or internal structure of particles (Friedrich, 2013). Therefore, the potential $V(r)$ in the Schrödinger differential equation (2.6.7) is generally a Coulomb potential combined with a short range potential.

The short-range potential will affect the outgoing wave in addition to the Coulomb potential. The incoming wave will be affected only by the coulomb potential (Friedrich, 2013). This tells that the incoming wave still be described by the Coulomb wave function whose the asymptotic behaviour is defined in the Eqn. (4.1.51). Therefore the new wave function, called also the modified Coulomb wave function will have an asymptotic behaviour described by

$$\psi(r) \approx \psi_c(r) + F_n(\theta, \phi)\frac{1}{r}e^{i(kr-\eta\ln(2kr))} \tag{4.1.54}$$

where $\psi_c(\theta)$ is the Coulomb wave-function obtained in the Eqn.(4.1.51) for spinless particles and $F_n(\theta, \phi)$ is the scattering amplitude due to the short-range potential.

To get the expression of $F_n(\theta, \phi)$, we consider the partial expansion of the pure Coulomb wave defined in the Eqn. (4.1.53).

The expansion of the partial the total wave function defined in the Eqn. (4.1.54) leads to

$$\psi(r) = \sum_{l=0}^{\infty}\frac{B_l(k)}{r}u_l(\vec{r})P_l(\cos\theta) \tag{4.1.55}$$

where in this case $u_l(\vec{r})$ is given by the global solution to the confluent Hyper-geometric Equation obtained in the Eqn. (4.1.50). Then, the factor

$$B_l(k)\frac{u_l(r)}{r} = \frac{B_l(k)}{r}\sin\left(kr - \eta\ln(2kr) - \frac{1}{2}l\pi + \sigma_l + \delta_l\right) \tag{4.1.56}$$

This can be written in term of incoming and outgoing wave as

$$B_l(k)\frac{u_l(r)}{r} = \frac{B_l(k)}{2ir}e^{i(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l+\delta_l)} - \frac{B_l(k)}{2ir}e^{-i(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l+\delta_l)} \tag{4.1.57}$$

Our goal is to find an expression for the constant $B_l(k)$. To the expression we equal the incoming wave of the first term of the Eqn. (4.1.53) and the incoming wave of the Eqn. (4.1.55). This gives

$$\frac{B_l(k)}{2ir}e^{-i(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l+\delta_l)} = \frac{e^{i\sigma_l}}{2ikr}(2l+1)\ i^l\ e^{-i(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l)} \tag{4.1.58}$$

There after, we can extract the constant $B_l(k)$ and get

$$B_l(k) = (2l+1)\ i^l\frac{e^{i(\sigma_l+\delta_l)}}{k} \tag{4.1.59}$$

With this, the full wave function (4.1.55) can be expanded asymptotically in the form

$$\psi(r) \simeq \sum_{l=0}^{\infty}(2l+1)\ i^l\ \frac{e^{i\sigma_l}e^{i\delta_{lj}}}{kr}\sin\left(kr - \eta\ln(2kr) - \frac{1}{2}l\pi + \sigma_l + \delta_l\right)\ P_l(\cos\theta) \tag{4.1.60}$$

The explicit expression of the scattered part $\psi_{sc}(r)$ of the Eqn. (4.1.54) contains the short-range scattering amplitude $F(\theta)$. To explicitly express this, we need to subtract the expansion of partial Coulomb wave obtained in the Eqn. (4.1.51) from the Eqn. (4.1.60) and get

$$\begin{aligned}\psi_{sc}(r) =&\frac{1}{kr}\sum_{l=0}^{\infty}(2l+1)\ i^l\ e^{i\sigma_l}\left[e^{i\delta_{lj}}\sin\left(kr - \eta\ln(2kr) - \frac{1}{2}l\pi + \sigma_l + \delta_{lj}\right)\right.\\ &\left.-\sin\left(kr - \eta\ln(2kr) - \frac{1}{2}l\pi + \sigma_l\right)\right]\ P_l(\cos\theta)\end{aligned} \tag{4.1.61}$$

Again, using the properties of the $\sin$ function in exponential form, we can write the r.h.s as

$$\begin{aligned}&e^{i\sigma_l}\left[e^{i\delta_{lj}}\sin\left(kr - \eta\ln(2kr) - \frac{1}{2}l\pi + \sigma_l + \delta_{lj}\right) - \sin\left(kr - \eta\ln(2kr) - \frac{1}{2}l\pi + \sigma_l\right)\right] =\\ &\frac{1}{2i}e^{i(\sigma_l+\delta_{lj})}\left[e^{i(\sigma_l+\delta_{lj})}e^{il\pi/2}e^{i(kr-\eta\ln(2kr))} - e^{-i(\sigma_l+\delta_{lj})}e^{-il\pi/2}e^{-i(kr-\eta\ln(2kr))}\right] -\\ &\frac{1}{2i}e^{i\sigma_l}\left[e^{i\sigma_l}e^{il\pi/2}e^{i(kr-\eta\ln(2kr))} - e^{-i\sigma_l}e^{-il\pi/2}e^{-i(kr-\eta\ln(2kr))}\right]\end{aligned} \tag{4.1.62}$$

Next we find

$$e^{i\sigma_l}\left[e^{i\delta_{lj}}\sin\left(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l+\delta_{lj}\right)-\sin\left(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l\right)\right]= \\ \frac{1}{2i}\left(e^{2i(\sigma_l+\delta_{lj})}e^{il\pi/2}e^{i(kr-\eta\ln(2kr))}-e^{-il\pi/2}e^{-i(kr-\eta\ln(2kr))}-e^{2i\sigma_l}e^{il\pi/2}e^{i(kr-\eta\ln(2kr))}+ \\ e^{-il\pi/2}e^{-i(kr-\eta\ln(2kr))}\right) \tag{4.1.63}$$

Some terms cancel out yielding

$$e^{i\sigma_l}\left[e^{i\delta_{lj}}\sin\left(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l+\delta_{lj}\right)-\sin\left(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l\right)\right]= \\ \frac{1}{2i}e^{2i(\sigma_l+\delta_{lj})}e^{il\pi/2}e^{i(kr-\eta\ln(2kr))}-\frac{1}{2i}e^{2i\sigma_l}e^{il\pi/2}e^{i(kr-\eta\ln(2kr))} \tag{4.1.64}$$

Finally we obtain for the transformed factor

$$e^{i\sigma_l}\left[e^{i\delta_{lj}}\sin\left(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l+\delta_{lj}\right)-\sin\left(kr-\eta\ln(2kr)-\frac{1}{2}l\pi+\sigma_l\right)\right]= \\ \frac{1}{2i}e^{2i\sigma_l}\left[e^{2i\delta_{lj}}-1\right]e^{il\pi/2}\ e^{i(kr-\eta\ln(2kr))} \tag{4.1.65}$$

As a result, the outgoing wave in the Eq. (4.1.61) can be rewritten as

$$\psi_{sc}(r)=\frac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)\ i^l\ e^{2i\sigma_l}\left(e^{2i\delta_{lj}`}-1\right)e^{il\pi/2}\ e^{i(kr-\eta\ln(2kr))}P_l(\cos\theta) \tag{4.1.66}$$

Since $i^l e^{-il\pi/2}=e^{l\ln i}e^{-il\pi/2}=e^{l\ln e^{i\pi/2}}e^{-il\pi/2}=e^{il\pi/2}e^{-il\pi/2}=1$, we can further simplify the result into

$$\psi_{sc}(r)=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\ e^{2i\sigma_{lj}}\ \left(e^{2i\delta_{lj}`}-1\right)\ P_l(\cos\theta)\ \frac{e^{i(kr-\eta\ln(2kr))}}{r} \tag{4.1.67}$$

By comparing with Eq.(4.1.54), we can deduce the expression of the matrix scattering amplitude as

$$F_n(\theta,\phi)=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\ e^{2i\sigma_{lj}}\ \left(e^{2i\delta_{lj}`}-1\right)\ P_l(\cos\theta) \tag{4.1.68}$$

This the scattering amplitude tied to the effect of the short-range potential. Altogether, in the asymptotic region the full wave function obtained in the Eqn.:(4.1.54) can be rewritten as

$$\psi(r) \approx e^{i(kz+\eta \ln 2k(r-z))} + \Big(F_c(\theta) + F_n(\theta,\phi)\Big)\frac{1}{r}e^{i(kr-\eta\ln(2kr))} \tag{4.1.69}$$

Using Eq. (4.1.28) and (4.1.68), the more explicit form becomes

$$\begin{aligned}\psi(r) \approx & e^{i(kz+\eta \ln 2k(r-z))} + \Bigg(-\frac{\eta\, e^{-i\eta\ln(\sin^2(\theta/2))+2\sigma_0}}{2k\sin^2(\theta/2)} + \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\, e^{2i\sigma_{lj}} \,\left(e^{2i\delta_{lj}{}^{\text{`}}} - 1\right) \\ & P_l(\cos\theta)\Bigg)\frac{1}{r}e^{i(kr-\eta\ln(2kr))}\end{aligned} \tag{4.1.70}$$

4.2 Matrix Coulomb scattering amplitude

In the case of spin particles, the scattering process is modelled in terms of a spin-angle Coulomb wave function $\psi(\vec{r},\sigma)$. As shown in the section 3.3, the scattering amplitude will be written as a Coulomb matrix in two dimensional space.

To get insights into the amplitude, let write the asymptotic form of the Coulomb wave function (4.1.70) incorporating the spin state of the particles:

$$\psi(\vec{r},\sigma) = \chi_{\frac{1}{2}m_s}(\sigma)e^{i(kz+\eta\ln 2k(r-z))} + \chi_{\frac{1}{2}m'_s}(\sigma)\left(M_{m'_s m_s} + \tilde{M}_{m'_s m_s}\right)\frac{e^{i(kr-\eta\ln 2kr)}}{r} \tag{4.2.1}$$

where $M_{m'_s m_s}$ and $\tilde{M}_{m'_s m_s}$ are respectively matrix Coulomb scattering amplitude and matrix scattering amplitude due to the short-range potential. This can be rewritten as

$$\psi(\vec{r},\sigma) \approx \chi_{\frac{1}{2}m_s}(\sigma)e^{i(kz+\eta\ln 2k(r-z))} + \chi_{\frac{1}{2}m'_s}(\sigma)\left[\delta_{(\frac{1}{2},m'_s)}F_c + \tilde{M}_{m'_s m_s}\right]\frac{e^{i(kr-\eta\ln 2kr)}}{r} \tag{4.2.2}$$

where $\delta_{(m_s,m'_s)}$ is a Kronecker delta which takes only two variable integers 1 or 0. It is defined by the fact that

$$\chi_{m_s,m'_s}(\sigma) = \begin{cases} 1 & \text{if } m'_s = m_s \\ 0 & \text{if } m'_s \neq m_s \end{cases} \tag{4.2.3}$$

The variable σ is different from the Pauli spin matrix, and it takes any value of the magnetic quantum number from s to $-s$. $m_s = \pm 1/2$, then $\delta_{(\frac{1}{2},\frac{1}{2})} = 1$ and $\delta_{(\frac{1}{2},-\frac{1}{2})} = 0$.

Then, we have explicitly

$$\psi(\vec{r},\sigma) \approx \chi_{\frac{1}{2}m_s}(\sigma)e^{i(kz+\eta\ln 2k(r-z))} + \Bigg[\chi_{\frac{1}{2}m_s}(\sigma)F_c(\theta) + \chi_{\frac{1}{2}m'_s}(\sigma)\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\, e^{2i\sigma_{lj}} \\ \left(e^{2i\delta_{lj}`} - 1\right)\; P_l(\cos\theta)\Bigg]\frac{1}{r}e^{i(kr-\eta\ln(2kr))} \tag{4.2.4}$$

since $P_l(cos\theta)$ is such that $m = 0$ and defined as

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\; Y_{l0}(\theta,\phi) \tag{4.2.5}$$

The asymptotic form (4.2.4) becomes

$$\psi(\vec{r},\sigma) \approx \chi_{\frac{1}{2}m_s}(\sigma)e^{i(kz+\eta\ln 2k(r-z))} + \Bigg[\chi_{\frac{1}{2}m_s}(\sigma)F_c(\theta) + \frac{1}{2ik}\sum_{l=0}^{\infty}\sqrt{4\pi(2l+1)}\; e^{2i\sigma_{lj}} \\ \left(e^{2i\delta_{lj}`} - 1\right)\; Y_{l0}(\theta,\phi)\chi_{\frac{1}{2}m'_s}(\sigma)\Bigg]\frac{1}{r}e^{i(kr-\eta\ln(2kr))} \tag{4.2.6}$$

In Eq. (3.3.11) we defined the product of the spin state and the spherical function as

$$Y_{l0}(\theta,\phi)\chi_{\frac{1}{2}m_s}(\sigma) = \sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\langle l0\frac{1}{2}m_s|jm\rangle\mathcal{Y}_{l\frac{1}{2}jm}(\theta,\phi,\sigma) \tag{4.2.7}$$

With this, the asymptotic form (4.2.6) can be rewritten as

$$\psi(\vec{r},\sigma) \approx \chi_{\frac{1}{2}m_s}(\sigma)e^{i(kz+\eta\ln 2k(r-z))} + \sum_{m'_s=\pm\frac{1}{2}}\Bigg[\chi_{\frac{1}{2}m_s}(\sigma)F_c(\theta) + \frac{1}{2ik}\sum_{l=0}^{\infty}\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sqrt{4\pi(2l+1)}\; e^{2i\sigma_{lj}} \\ \left(e^{2i\delta_{lj}`} - 1\right)\; \langle l0\frac{1}{2}m_s|jm\rangle\mathcal{Y}_{l\frac{1}{2}jm}(\theta,\phi,\sigma)\Bigg]\frac{1}{r}e^{i(kr-\eta\ln(2kr))} \tag{4.2.8}$$

From the Eq. (2.4.9) the spherical eigenfunction $\mathcal{Y}_{lm_l\frac{1}{2}m_s}(\theta,\phi)$ for particle with spin and orbit angular momentum are given by

$$\mathcal{Y}_{lm_l\frac{1}{2}m_s}(\theta,\phi,\sigma) = \sum_{m_l\, m_s}\langle lm_l\frac{1}{2}m_s|jm\rangle Y_{lm_l}(\theta,\phi)\chi_{\frac{1}{2}m_s}(\sigma) \tag{4.2.9}$$

where $m = m_s$ and $m = m_l + m'_s$, i.e. $m_l = m_s - m'_s$. so that

$$\psi(\vec{r},\sigma) \approx \chi_{\frac{1}{2}m_s}(\sigma)e^{i(kz+\eta\ln 2k(r-z))} + \Bigg[\chi_{\frac{1}{2}m_s}(\sigma)F_c(\theta) + \frac{1}{2ik}\sum_{m'_s=\pm\frac{1}{2}}\sum_{l=0}^{\infty}\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sqrt{4\pi(2l+1)}\; e^{2i\sigma_{lj}}$$
$$\left(e^{2i\delta_{lj}`} - 1\right)\; \langle l0\frac{1}{2}m_s|jm_s\rangle\langle lm_s - m'_s\frac{1}{2}m'_s|jm_s\rangle Y_{lm_s-m'_s}(\theta,\phi)\chi_{\frac{1}{2}m_s}(\sigma)\Bigg]\frac{1}{r}e^{i(kr-\eta\ln(2kr))} \tag{4.2.10}$$

Comparing the above result to that in Eq. (4.2.1) we obtain that the matrix scattering amplitude as

$$M_{m'_s m_s} = \delta_{(m'_s,m'_s)}F_c(\theta) + \frac{1}{2ik}\sum_{l=0}^{\infty}\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sqrt{4\pi(2l+1)}\; e^{2i\sigma_l}\left[e^{2i\delta_{lj}} - 1\right]$$
$$\langle l0\frac{1}{2}m_s|jm_s\rangle\langle lm_s - m'_s\frac{1}{2}m'_s|jm_s\rangle\; Y_{lm_s-m'_s}(\theta,\phi) \tag{4.2.11}$$

where $\langle l0\frac{1}{2}m_s|jm_s\rangle$ and $\langle lm_s - m'_s\frac{1}{2}m'_s|jm_s\rangle$ are the Clebsch-Gordan coefficient and $Y_{lm_s-m'_s}(\theta,\phi)$ is the spherical harmonics whose their explicit form is given in the Appendices (B) and (C).

4.3 DCS for spin and charged particles

The differential cross section for incident particles with quantum number k and spin state m_s elastically scattered by a spin-zero target, into angle (θ,ϕ) and a spin state m'_s is given by

$$\frac{d\sigma}{d\Omega}(\theta,\phi) = \left|\delta m_s, m'_s F_c(\theta) + \tilde{M}_{m'_s m_s}(\theta,\phi)\right|^2 \tag{4.3.1}$$

and cross section for scattering of unpolarized into angle θ is

$$\frac{d\sigma}{d\Omega}(\theta) = \frac{1}{2s+1}\sum_{m_s,m'_s}\left|\delta m_s, m'_s F_c(\theta) + \tilde{M}_{m'_s m_s}(\theta,\phi)\right|^2 \tag{4.3.2}$$

We know that, for two given complex numbers z_1 and z_2, the norm $|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2\mathcal{R}e(z_1^* z_2)$ Substituting the expression (4.2.11) into (4.3.1), we have

$$\frac{d\sigma}{d\Omega}(\theta,\phi) = \sum_{m_s,m'_s}\frac{1}{2s+1}\left[|\delta_{(\frac{1}{2},m'_s)}F_c(\theta)|^2 + |\tilde{M}(\theta)_{m'_s m_s}|^2 + 2\mathcal{R}e\left((F_c)^*\;\tilde{M}_{m'_s m_s}\right)\right] \tag{4.3.3}$$

where $F_c(\theta)$ is the Coulomb scattering amplitude. With

$$|F_c(\theta)|^2 = \left|-\frac{\eta\; e^{-i\eta\ln(\sin^2(\theta/2))+2i\sigma_0}}{2k\sin^2(\theta/2)}\right|^2 = \frac{\eta^2}{4k^2\sin^4(\theta/2)} = \frac{(Z_i Z_t)^2}{16E^2\sin^4(\theta/2)} \tag{4.3.4}$$

and

$$|\tilde{M}_{m_s',m_s}|^2 = \Bigg[\frac{1}{4k^2}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sqrt{4\pi(2l+1)}\sqrt{4\pi(2l'+1)}\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sum_{j=|l'-\frac{1}{2}|}^{|l'+\frac{1}{2}|}\left(e^{21\sigma_l}\right)\left(e^{2i\sigma_{l'}}\right)^* \\ \left(e^{2i\delta_{lj}}-1\right)\left(e^{2i\delta_{l'j'}}-1\right)^*\sum_{m_s,m_s'}\langle l0\frac{1}{2}m_s|jm_s\rangle\langle lm_s-m_s'\frac{1}{2}m_s'|jm_s\rangle \\ \langle l'0\frac{1}{2}m_s|jm_s\rangle\langle l'm_s-m_s'\frac{1}{2}m_s'|jm_s\rangle\, Y_{lm_s-m_s'}(\theta,\phi)\, Y^*_{lm_s-m_s'}(\theta,\phi)\Bigg] \tag{4.3.5}$$

For an incident particle with initial spin state along $\vec{k}$: $m_s=\frac{1}{2}$, and scattered with final spin state in the same direction : $m_s'=\frac{1}{2}$, the differential cross section is given by

$$\left.\frac{d\sigma}{d\Omega}(\theta,\phi)\right|_{\uparrow\uparrow} = \left[|F_c(\theta)|^2+|\tilde{M}(\theta)_{\uparrow\uparrow}|^2+2\mathcal{R}e\left((F_c)(\theta)^*\,\tilde{M}(\theta)_{\uparrow\uparrow}\right)\right] \tag{4.3.6}$$

where $\left|\tilde{M}(\theta)_{\uparrow\uparrow}\right|^2$ is given by

$$\left|\tilde{M}(\theta)_{\uparrow\uparrow}\right|^2 = \Bigg[\frac{1}{4k^2}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sqrt{4\pi(2l+1)}\sqrt{4\pi(2l'+1)}\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sum_{j=|l'-\frac{1}{2}|}^{|l'+\frac{1}{2}|}\left(e^{21\sigma_l}\right)\left(e^{2i\sigma_{l'}}\right)^* \\ \left(e^{2i\delta_{lj}}-1\right)\left(e^{2i\delta_{l'j'}}-1\right)^*\langle l0\frac{1}{2}\frac{1}{2}|j\frac{1}{2}\rangle\langle l0\frac{1}{2}\frac{1}{2}|j\frac{1}{2}\rangle\;\langle l'0\frac{1}{2}\frac{1}{2}|j\frac{1}{2}\rangle\langle l'0\frac{1}{2}\frac{1}{2}|j\frac{1}{2}\rangle \\ Y_{l0}(\theta,\phi)\; Y^*_{l0}(\theta,\phi)] \tag{4.3.7}$$

For an incident particle with initial spin along $\vec{k}$: $m_s=\frac{1}{2}$, and scattered with final spin state in opposite: $m_s'=-\frac{1}{2}$, the direct Coulomb contribution disappears, therefore the differential cross section is given by

$$\left.\frac{d\sigma}{d\Omega}(\theta,\phi)\right|_{\uparrow\downarrow} = \left|\tilde{M}(\theta)_{\uparrow\downarrow}\right|^2 \tag{4.3.8}$$

where $\left|\tilde{M}(\theta)_{\uparrow\downarrow}\right|^2$ is given by

$$\left|\tilde{M}(\theta)_{\uparrow\downarrow}\right|^2 = \Bigg[\frac{1}{4k^2}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sqrt{4\pi(2l+1)}\sqrt{4\pi(2l'+1)}\sum_{j=|l-\frac{1}{2}|}^{|l+\frac{1}{2}|}\sum_{j=|l'-\frac{1}{2}|}^{|l'+\frac{1}{2}|}\left(e^{21\sigma_l}\right)\left(e^{2i\sigma_{l'}}\right)^* \\ \left(e^{2i\delta_{lj}}-1\right)\left(e^{2i\delta_{l'j'}}-1\right)^*\langle l'0\frac{1}{2}\frac{1}{2}|j\frac{1}{2}\rangle\langle l'1\frac{1}{2}-\frac{1}{2}|j\frac{1}{2}\rangle\; Y_{l1}(\theta,\phi)\; Y^*_{l1}(\theta,\phi)\Bigg] \tag{4.3.9}$$

4.4 Analyzing power

The expression for the analyzing power, denoted by A_y is provided in Eq. (2.6.30), and we recall it as

$$A_y = \frac{2\, Im\left[f(k,\theta)\cdot g(k,\theta)^*\right]}{|f(k,\theta)|^2 + |g(k,\theta)|^2} \tag{4.4.1}$$

Employing Eq. (3.3.28) - (3.3.33) in Eq. (4.2.11) we obtain

$$f(k,\theta)\Big|_{\frac{1}{2},\frac{1}{2}} = F_c(\theta) + \frac{1}{2ik}\sum_{l=0}^{\infty}\left[(l+1)\left(e^{2i\delta_{l,l+\frac{1}{2}}(k)} - 1\right) + l\left(e^{2i\delta_{l,l-\frac{1}{2}}(k)} - 1\right)\right] P_l(\cos\theta) \tag{4.4.2}$$

above is the direct scattering amplitude in the case of a particle.

and

$$g(k,\theta) = \frac{1}{2ik}\sum_{l=1}^{\infty}\left(e^{2i\delta_{l,l+\frac{1}{2}}(k)} - e^{2i\delta_{l,l-\frac{1}{2}}(k)}\right) P_l^{(1)}(\cos\theta) \tag{4.4.3}$$

the spin-flip scattering amplitude. $P_l(\cos\theta)$ is the Legendre polynomial functions whose explicit expressions are given in the Appendix (B.1).

Since $f(\theta)$ and $g(\theta)$ are complex functions, their moduli squared are

$$\begin{aligned}
|f(\theta)|^2 =& \frac{(Z_i Z_t)^2}{16E^2\sin^4(\theta/2)} + \frac{1}{4k^2}\left\{\sum_{l=0}^{\infty} e^{2i\sigma_l}\left[(l+1)\left(e^{2i\delta_{l,l+\frac{1}{2}}} - 1\right) + l\left(e^{2i\delta_{l,l-\frac{1}{2}}} - 1\right)\right]\right. \\
&\left.\sum_{l=0}^{\infty}\left(e^{2i\sigma_l}\right)^*\left[(l+1)\left(e^{2i\delta_{l,l+\frac{1}{2}}} - 1\right) + l\left(e^{2i\delta_{l,l-\frac{1}{2}}} - 1\right)\right]^*\right\}(P_l(\cos\theta))^2 \\
&+ 2\mathcal{R}e\left(-\frac{\eta\, e^{i\eta\ln(\sin^2(\theta/2)) - 2i\sigma_0}}{2k\sin^2(\theta/2)}\right)\frac{1}{2ik}\sum_{l=0}^{\infty} e^{2i\sigma_l}\left((l+1)\left(e^{2i\delta_{l,l+\frac{1}{2}}} - 1\right)\right. \\
&\left. + l\left(e^{2i\delta_{l,l-\frac{1}{2}}} - 1\right)\right) P_l(\cos\theta)
\end{aligned} \tag{4.4.4}$$

and

$$|g(\theta)|^2 = \frac{1}{4k^2}\left\{\sum_{l=1}^{\infty} e^{2i\sigma_l}\left[e^{2i\delta_{l,l+\frac{1}{2}}} - e^{2i\delta_{l,l-\frac{1}{2}}}\right]\sum_{l=0}^{\infty}\left(e^{2i\sigma_l}\right)^*\left[e^{2i\delta_{l,l+\frac{1}{2}}} - e^{2i\delta_{l,l-\frac{1}{2}}}\right]^*\right\}(P_l^{(1)}(\cos\theta))^2 \tag{4.4.5}$$

These scattering amplitudes $f(k,\theta)$ and $g(k,\theta)$ are either predicted or deduced in a scattering experiment. During the scattering process, $f(\theta)$ is related to the invariance of spin direction and $g(\theta)$ describes the change of spin direction.(Kessler, 1969)

5. Conclusion

Scattering experiments are central to the work of nuclear physicists, and this is demonstrated by a volume of research and publication in this field and in the growth of the quantity of data on nucleonic scattering throughout the world. Our aim in this project was to arrive at the formulas allowing to compute the values of some observable parameters that can be measured during the scattering process in laboratory. We discussed the scattering following concepts of Classical and Quantum mechanics and we made use of some mathematical tools in solving of partial differential equation.

The motion of particles, has been modelled by the time-independent Schroödinger differential equation 2.6.7. We considered intrinsic orbital and spin of particles coming far away toward a target and going far away from it. We have shown that the asymptotic form of the solution to the time-independent Schrödinger differential equation can be represented as the sum of incoming and outgoing wave-functions 2.6.8. We expressed the same solution in partial wave decomposition as the sum of product of radial and spherical harmonic function 3.1.2.

Comparison between the asymptotic and partial wave forms of the solution was very useful in extracting the expression for the scattering amplitude 3.1.24 which is one of the goal in tending expressions for the differential cross section and analyzing power. For spinless neutral particles, only the cross section can be computed 3.2.1. For charged and spinless particles, e.g. alpha particles scattered with nuclei, the Coulomb interaction acts as a long range scattering potential 4.1.28 combined with a short range potential 4.1.68. In this case, the scattering process is modelled in terms of the Confluent Hyper-geometric equation called also the Kummer's equation of the first kind 4.1.36 whose solution is the Confluent Hyper-geometric function 4.1.37. This kind of solution helped to express again the partial wave function 4.1.60 and its asymptotic form 4.1.69, in order to find scattering amplitudes 4.1.68 and 4.1.28.

For charged particles with spin, e.g. protons scattered off nuclei, we included the concept of spin state, by expressing the scattering amplitude in terms of a 2×2 matrix 4.2.11. The Coulomb and nuclear potentials appeared in the direct amplitude scattering 4.4.2 when there is no change in spin state of the particle during the process. For change in spin direction, only the nuclear amplitude 4.4.3 appears. We expressed again the differential cross section 4.3.3, and found the possibility of expressing the analyzing power 4.4.1 in order to quantify polarization effects of particles during the scattering process.

A future plan is to apply the expressions 4.3.3 and 4.4.1 in practice. Experimental differential cross section and the analyzing power, function of the scattering angle, measured from the experiments will be compared to the theoretical values in order to conclude about validity of theoretical assumptions about short-range interactions and to predict the outcomes of a future measurements.

Appendices

AppendixA. Angular momentum

A.1 Properties of Hermitian operators

Consider a vector operator of angular momentum J. Its component satisfy the commutation relations: (Messiah, 1962)

$$[J_x, J_y] = iJ_z; \; [J_y, J_z] = iJ_x; \text{ and } [J_z, J_x] = iJ_y \tag{A.1.1}$$

The square of J is

$$J^2 = J_x^2 + J_y^2 + J_z^2 \tag{A.1.2}$$

J^2 can commute with J_x, J_y and J_z. In general, it will also commute with any function of the components of J.

We define the Hermitian conjugate operators, denoted as J_+ and J_- with

$$J_+ = J_x + iJ_y, \; J_- = J_x - iJ_y, \text{ and } J_+ = J_-^+ \tag{A.1.3}$$

These satisfy the following relations:

$$[J_z, J_\pm] = \pm J_\pm; \text{ and } [J_x, J_-] = 2J_z \tag{A.1.4}$$

As J^2 commute with any function of the components of J, we can write also

$$[J^2, J_+] = [J^2, J_-] = [J^2, J_z] = 0 \tag{A.1.5}$$

From Eq. (A.1.2), we find

$$J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2 \tag{A.1.6}$$

A.2 Eigenstates of J_z and J

Let $|jm\rangle$ be an eigenvector of J^2 corresponding to the eigenvalues of $j(j+1)$ and m, respectively. This means that acting J^2 and J_z on $|jm\rangle$ gives (Messiah, 1962)

$$J^2|jm\rangle = j(j+1)|jm\rangle \text{ and } J_z|jm\rangle = m|jm\rangle \tag{A.2.1}$$

Exploiting the properties of the Hermitian operators defined above, applying simultaneously J_- and J_+ on the eigenvector $|jm\rangle$ we get

$$J_{\mp}J_{\pm}|jm\rangle = [j(j \pm 1) - m(m \pm 1)]\,|jm\rangle \tag{A.2.2}$$

In a similar way, we get

$$J_{\mp}J_{\pm}|jm\rangle = [(j \mp m)(j \pm m \mp 1)]\,|jm\rangle \tag{A.2.3}$$

The norm of the vectors $J_+|jm\rangle$ and $J_-|jm\rangle$ is

$$\langle jm|J_{\mp}J_{\pm}|jm\rangle = (j \mp m)(j \pm m \mp 1)\langle|jm\rangle \tag{A.2.4}$$

Since $(j \mp m)(j \pm m \mp 1) \geq 0$, this implies that

$$-j \leq m \leq j \tag{A.2.5}$$

Note that, the only posible values of m are $(2j + 1)$.

A.3 Orbital angular momentum L_z

Consider a particle of mass M, a momentum $\vec{p}$ at a position $\vec{r}$. From classical mechanics point of view, the particle orbital angular momentum $\vec{L}$ is defined by

$$\vec{L} = \vec{r} \times \vec{p} \tag{A.3.1}$$

In cartesian coordinates, the components of $\vec{L}$ are

$$L_x = yp_z - zp_y;\; L_y = zp_x - xp_z;\; \text{and } L_z = xp_y - yp_x \tag{A.3.2}$$

Quantally, the operators

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right);\; L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right);\; \text{and } L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) \tag{A.3.3}$$

These can be seen as following from the vector equation

$$\vec{L} = -i\hbar(\vec{r} \times \vec{\nabla}) \tag{A.3.4}$$

The square of the orbital momentum is

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

.

In spherical coordinates, related to the cartesian coordinates, we have

$$x = r\sin\theta\cos\phi; \;\; y = r\sin\theta\sin\phi \text{ and } z = r\cos\theta \tag{A.3.5}$$

From the Eq. (A.3.3), we find

$$L_x = -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \coth\theta\cos\phi\frac{\partial}{\partial\phi}\right) \tag{A.3.6}$$

$$L_y = -i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} - \coth\theta\sin\phi\frac{\partial}{\partial\phi}\right) \tag{A.3.7}$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi} \tag{A.3.8}$$

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{A.3.9}$$

A.4 Simultaneous eigenfucntion of L_z and L^2

The eigenvalue of L_z is denoted by $m\hbar$. The corresponding eigenfunctions must have a separable dependence on ϕ since L_z depends only on ϕ. Let us denote the eigenfunction as $\Phi(\phi)$. We can write the eigenvalue equation as like

$$L_z\Phi(\phi) = m\hbar\Phi(\phi) \tag{A.4.1}$$

Using L_z in spherical coordinates we can write

$$-i\hbar\frac{\partial}{\partial\phi}\Phi(\phi) = m\hbar\Phi(\phi) \tag{A.4.2}$$

which is a differential equation whose the solution gives the eigenfunction of L_z as

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi} \tag{A.4.3}$$

From Eq. (A.3.6) - (A.4.3), we can see that the simultaneous eigenfunctions of L^2 and L_z depend on θ and ϕ. As the eigenvalues of the L^2 are given by $l(l+1)\hbar^2$, we can write

$$L^2 Y_{lm}(\theta,\phi) = l(l+1)\hbar^2 Y_{lm}(\theta,\phi) \tag{A.4.4}$$

$$L_z Y_{lm}(\theta,\phi) = m\hbar Y_{lm}(\theta,\phi) \tag{A.4.5}$$

Using the form of L^2 in spherical coordinates, we can write

$$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{lm}(\theta,\phi)=l(l+1)\hbar^2Y_{lm}(\theta,\phi) \tag{A.4.6}$$

We can then write

$$Y_{lm}(\theta,\phi)=\Theta_{lm}(\theta)\Phi_m(\phi) \tag{A.4.7}$$

where $\Phi(\phi)$ is given in the Eq. (A.4.3).

Thereafter $\Theta(\theta)$ is obtained by solving the differential equation

$$\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]\right\}\Theta(\theta)=0 \tag{A.4.8}$$

Solving the equation, we change the variable to $\cos\theta=\omega$. Then the differential Eqn. (A.4.8) becomes

$$\left[(1-\omega^2)\frac{d^2}{d\omega^2}-2\omega\frac{d}{d\omega}+l(l+1)-\frac{m^2}{1-\omega^2}\right]\Theta_{lm}(\omega)=0 \tag{A.4.9}$$

This is the kind of Legendre's associated differential equation and the solution is the Legendre's associated functions defined by

$$P_l^m(\omega)=(1-\omega^2)^{|m|/2}\left(\frac{d}{d\omega}\right)^{|m|}P_l(\omega) \tag{A.4.10}$$

where $P_l(\omega)$ is the Legendre polynomial defined by

$$P_l(\omega)=\frac{1}{2^l\ l!}\left(\frac{d}{d\omega}\right)^l(\omega^2-1)^l \tag{A.4.11}$$

The corresponding solution to the Eqn. (A.4.8) is

$$\Theta_{lm}(\theta)=\begin{cases}(-1)^m\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}P_l^m(\cos\theta)\text{ for }m\geq 0\\[2ex](-1)^m\Theta^*_{lm}(\theta)\text{ for }m<0\end{cases} \tag{A.4.12}$$

Following Eq. (A.4.3) and Eq. (A.4.7), we arrive at the expressions for simultaneous eigenfunctions of L^2 and L_z in the form

$$Y_{lm}(\theta,\phi)=\begin{cases}(-1)^m\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}P_l^m(\cos\theta)e^{im\phi}\text{ for }m\geq 0\\[2ex](-1)^mY^*_{lm}(\theta,\phi)\text{ for }m<0\end{cases} \tag{A.4.13}$$

AppendixB. Legendre polynomial

B.1 Explicit form of the Legendre polynomial

The Legendre polynomial is given by the Eqn. (A.4.11) called the Rodrigues formula. The explicit results are

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$
8	$\frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$
9	$\frac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)$
10	$\frac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)$

Figure B.1: Explicit form of the Legendre polynomials.

B.2 Explicit form of the associated Legendre functions

The associated Legendre functions are given by the Eqn. (A.4.10). The explicit forms of the lowest functions are

$$P_1^1(x) = (1-x^2)^{1/2},$$
$$P_2^1(x) = 3x(1-x^2)^{1/2},$$
$$P_2^2(x) = 3(1-x^2),$$
$$P_3^1(x) = \frac{3}{2}(5x^2-1)(1-x^2)^{1/2},$$
$$P_3^2(x) = 15x(1-x^2),$$
$$P_3^3(x) = 15(1-x^2)^{3/2},$$
$$P_4^1(x) = \frac{5}{2}(7x^3-3x)(1-x^2)^{1/2},$$
$$P_4^2(x) = \frac{15}{2}(7x^2-1)(1-x^2),$$
$$P_4^3(x) = 105x(1-x^2)^{3/2},$$
$$P_4^4(x) = 105(1-x^2)^2.$$

Figure B.2: Explicit forms of the Associated Legendre functions.

B.3 Explicit form of the spherical harmonics

The spherical harmonic functions is given in Eq. (A.4.13) and their explicit forms are

$$Y_{00}(\theta,\phi)=\left(\frac{1}{4\pi}\right)^{1/2}, \qquad Y_{2\pm1}(\theta,\phi)=\mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta\exp(\pm i\phi),$$

$$Y_{10}(\theta,\phi)=\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta, \qquad Y_{2\pm2}(\theta,\phi)=\left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta\exp(\pm 2i\phi),$$

$$Y_{1\pm1}(\theta,\phi)=\mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\exp(\pm i\phi), \qquad Y_{30}(\theta,\phi)=\left(\frac{7}{16\pi}\right)^{1/2}(5\cos^3\theta-3\cos\theta),$$

$$Y_{20}(\theta,\phi)=\left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta-1), \qquad Y_{3\pm1}(\theta,\phi)=\mp\left(\frac{21}{64\pi}\right)^{1/2}\sin\theta(5\cos^2\theta-1)\exp(\pm i\phi),$$

Figure B.3: Explicit forms of the spherical harmonic functions.

AppendixC. Explicit expressions of the Clebsch-Gordan

We have seen in Chapter 1, that an angular momentum eigenstate can be converted by a unitary transformation from one basis to other basis with

$$|j_1 j_2 J\, M\rangle = \sum_{m_1\, m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2 | j_1 j_2 J\, M\rangle \tag{C.0.1}$$

Consider two eigenfunctions $\psi_{j_1 m_1}(1)$ and $\psi_{j_2 m_2}(2)$ of the operators J_1^2, $J_{z,1}$, J_2^2 and $J_{z,2}$. Using the unitary transformation we can write a new eigenfunction $\psi_{j_1 m_1 j_2 m_2}(1,2)$ as

$$\psi_{j_1 m_1 j_2 m_2}(1,2) = \sum_{m_1\, m_2} \langle j_1 m_1 j_2 m_2 | j_1 j_2 J\, M\rangle \psi_{j_1 m_1}(1)\psi_{j_2 m_2}(2) \tag{C.0.2}$$

Alternatively, we can write

$$\psi_{j_1 m_1 j_2 m_2}(1,2) = \sum_{m_1\, m_2} \langle j_1 m_1 j_2 m_2 | J\, M\rangle \psi_{j_1 m_1}(1)\psi_{j_2 m_2}(2) \tag{C.0.3}$$

where $\langle j_1 m_1 j_2 m_2 | J\, M\rangle$ is the Clebsch-Gordan coefficients defined in the Eq. (2.4.10). These coefficients satisfy the orthogonality relations:

$$\sum_{m_1\, m_2} \langle j_1 m_1 j_2 m_2 | J\, M\rangle \langle j_1' m_1' j_2' m_2' | J\, M\rangle = \delta_{m_1, m_1'}\delta_{m_2, m_2'} \tag{C.0.4}$$

From this orthogonality relation, Eq. (C.0.3) can be inverted to yield

$$\psi_{j_1 m_1}(1)\psi_{j_2 m_2}(2) = \sum_{j} \langle j_1 m_1 j_2 m_2 | J\, M\rangle \psi_{j_1 m_1 j_2 m_2}(1,2) \tag{C.0.5}$$

The explicit form of this Clebsch-Gordan coefficients of interest for us can be found in Fig. (C.1)

j	$m_2 = 1/2$	$m_2 = -1/2$
$j_1 + \frac{1}{2}$	$\left[\frac{j_1 + m + \frac{1}{2}}{2j_1 + 1}\right]^{1/2}$	$\left[\frac{j_1 - m + \frac{1}{2}}{2j_1 + 1}\right]^{1/2}$
$j_1 - \frac{1}{2}$	$-\left[\frac{j_1 - m + \frac{1}{2}}{2j_1 + 1}\right]^{1/2}$	$\left[\frac{j_1 + m + \frac{1}{2}}{2j_1 + 1}\right]^{1/2}$

Figure C.1: Explicite expression of the Clebsch-Gordan.

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